geometry

Article

Problems in Invariant Differential Operators on Homogeneous Manifolds

Jae-Hyun Yang

MDPI

https://doi.org/10.3390/geometry2020009

geometry

MDPI

Article

Problems in Invariant Differential Operators on Homogeneous Manifolds

Jae-Hyun Yang [1,2]

[1] Yang Institute for Advanced Study, Hyundai 41 Tower, No. 1905, 293 Mokdongdong-ro, Yangcheon-gu, Seoul 07997, Republic of Korea; jhyang@inha.ac.kr or jhyang8357@gmail.com
[2] Department of Mathematics, Inha University, Incheon 22212, Republic of Korea

Abstract: In this paper, we consider six homogeneous manifolds $GL(n,\mathbb{R})/O(n,\mathbb{R})$, $SL(n,\mathbb{R})/SO(n,\mathbb{R})$, $Sp(2n,\mathbb{R})/U(n)$, $(GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)})/O(n,\mathbb{R})$, $(SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)})/SO(n,\mathbb{R})$, $(Sp(2n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)})/(U(n) \times S(m,\mathbb{R}))$. They are homogeneous manifolds which are important geometrically and number theoretically. These first three spaces are well-known symmetric spaces and the other three are not symmetric spaces. It is well known that the algebra of invariant differential operators on a symmetric space is commutative. The algebras of invariant differential operators on these three non-symmetric spaces are not commutative and have complicated generators. We discuss invariant differential operators on these non-symmetric spaces and provide natural but difficult problems about invariant theory.

Keywords: invariant differential operators; homogeneous manifolds; symmetric spaces; automorphic forms

MSC: primary 53C30; 53C35; 11F55

Academic Editor: Ion Mihai

Received: 14 December 2024
Revised: 8 April 2025
Accepted: 6 May 2025
Published: 9 June 2025

Citation: Yang, J.-H. Problems in Invariant Differential Operators on Homogeneous Manifolds. *Geometry* **2025**, *2*, 9. https://doi.org/10.3390/geometry2020009

1. Introduction

We consider the following six homogeneous manifolds which are important geometrically and number theoretically. We list them below.

(I) $GL(n,\mathbb{R})/O(n,\mathbb{R})$;
(II) $SL(n,\mathbb{R})/SO(n,\mathbb{R})$;
(III) $Sp(2n,\mathbb{R})/U(n)$;
(IV) $(GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)})/O(n,\mathbb{R})$;
(V) $(SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)})/SO(n,\mathbb{R})$;
(VI) $(Sp(2n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)})/(U(n) \times S(m,\mathbb{R}))$.

Here, $H_{\mathbb{R}}^{(n,m)}$ is the Heisenberg group defined by Formula (8) and $S(m,\mathbb{R})$ denotes the additive group consisting of all $m \times m$ real symmetric matrices. The above three (I), (II), and (III) are symmetric spaces of real dimension $\frac{n(n+1)}{2}$, $\frac{n(n+1)}{2} - 1$ and $n(n+1)$, respectively. In particular, the symmetric space (III) is an Einstein–Kähler Hermitian symmetric manifold. The theory of automorphic forms on these spaces was developed by Selberg, Maass, Siegel, and outstanding number theorists. The adelic version of the theory of automorphic forms on these spaces was developed by the Langlands school. It is well known that the algebras $\mathbb{D}$(I), $\mathbb{D}$(II) and $\mathbb{D}$(III) of all invariant differential operators on these three symmetric spaces, respectively, are finitely generated and commutative. Those algebras are polynomial algebras. A set of algebraic independent generators of $\mathbb{D}$(I) was first found by Maass and Selberg explicitly (cf. [1–3]).Later, Helgason [4] provided another set of algebraically independent generators of $\mathbb{D}$(I). A set of explicit algebraic independent generators of $\mathbb{D}$(II) was recently constructed by Brennecken, Ciardo, and Hilgert [5] using the so-called Maass–Selberg operators. An explicit set of algebraic independent generators of $\mathbb{D}$(III) was found by Maass (cf. [2]).

The remaining three (IV), (V), and (VI) are not symmetric homogeneous manifolds of real dimension $\frac{n(n+1)}{2} + mn$, $\frac{n(n+1)}{2} + mn - 1$ and $n(n+1) + 2\,mn$, respectively. The homogeneous space (VI) is a Kähler manifold and so is a symplectic manifold. The theory of automorphic forms including Jacobi forms on the homogeneous space (VI) was developed in the past three decades but is still not well established. So far, nobody has developed the theory of automorphic forms on the homogeneous spaces (IV) and (V). The algebras $\mathbb{D}$(IV), $\mathbb{D}$(V), and $\mathbb{D}$(VI) of all invariant differential operators on these three non-symmetric homogeneous spaces, respectively, are not commutative. Recently, it was shown that $\mathbb{D}$(IV), $\mathbb{D}$(V), and $\mathbb{D}$(VI) are finitely generated. So, any set of a finite system of generators of each of the algebras $\mathbb{D}$(IV), $\mathbb{D}$(V), and $\mathbb{D}$(VI) has algebraic relations (the Second Funddamental Theorem of Invariant Theory). Unfortunately, nobody found explicit generators of algebras $\mathbb{D}$(IV), $\mathbb{D}$(V), and $\mathbb{D}$(VI) as of now.

The aim of this article is to study invariant differential operators on the homogeneous manifolds (IV), (V), and (VI) and provide some problems of the classical invariant theory. The paper is organized as follows. In Section 2, we briefly review some properties on differential operators on homogeneous manifolds following Chapter II of Helgason's book [4]. In Section 3, we review $GL(n,\mathbb{R})$-invariant differential operators on the symmetric space (I) following the works of Selberg and Maass. We note that the symmetric space (I) is diffeomorphic to the open convex cone $\mathscr{P}_n$ in the Euclidean space $\mathbb{R}^N$ with $N = \frac{n(n+1)}{2}$ given by

$$\mathscr{P}_n := \{\, Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \,\}. \tag{1}$$

We providethe notion of automorphic forms on $\mathscr{P}_n$ defined by Selberg, Maass, and Terras using the algebra $\mathbb{D}$(I) of invariant differential operators on $\mathscr{P}_n$ (cf. [6], p. 234). In Section 4, we review $SL(n,\mathbb{R})$-invariant differential operators on the symmetric space (II). We note that the symmetric space (II) is diffeomorphic to the following symmetric space:

$$\mathfrak{P}_n := \{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0,\ \det(Y) = 1 \,\}. \tag{2}$$

We provide the notion of Maass forms on $\mathfrak{P}_n$ defined by Goldfeld [7] (Definition 5.1.3, pp. 115–116). In Section 5, we review $Sp(2n,\mathbb{R})$-invariant differential operators on the symmetric space (III). Maass found the explicit algebraically independent generators of $\mathbb{D}$(III) (cf. [2], pp. 112–118) and Shimura [8] also found algebraically independent generators of $\mathbb{D}$(III) using the universal enveloping algebra of the Lie algebra of the symplectic group $Sp(2n,\mathbb{R})$. We note that the symmetric space (III) is biholomorphic to the so-called Siegel upper half plane $\mathbb{H}_n$ given by

$$\mathbb{H}_n := \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im}\Omega > 0 \,\}. \tag{3}$$

It is known that $\mathbb{H}_n$ is an Einstein–Kähler Hermitian manifold of complex dimension $\frac{n(n+1)}{2}$ which is biholomorphic to the generalized unit disk $\mathbb{D}_n$ given by

$$\mathbb{D}_n := \{\, W \in \mathbb{C}^{(n,n)} \mid W = {}^tW,\ I_n - W\overline{W} > 0 \,\}. \tag{4}$$

The symmetric complex manifold $\mathbb{H}_n$ provides the rich, deep, and beautiful theory in algebraic geometry (e.g., Satake compactification, toroidal compactifications, moduli of abelian varieties, etc.) and number theory (e.g., Siegel modular forms, L-functions, etc.). We provide the notion of Maass–Siegel functions using the algebra $\mathbb{D}(\mathbb{H}_n)$. We also provide the notion of Siegel–Maass forms defined by Kramer and Mandal (cf. [9]). In Section 6, we study $GL_{n,m}(\mathbb{R})$-invariant differential operators on the non-symmetric homogeneous space (IV). We show that the semidirect product $GL_{n,m}(\mathbb{R})$ of $GL(n,\mathbb{R})$ and the additive group $\mathbb{R}^{(m,n)}$ given by

$$GL_{n,m}(\mathbb{R}) := GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

acts naturally and transitively on the following space:

$$\mathscr{P}_{n,m} := \mathscr{P}_n \times \mathbb{R}^{(m,n)} = \{\, Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \,\} \times \mathbb{R}^{(m,n)}. \tag{5}$$

We refer to Formula (61) for the precise action of $GL_{n,m}(\mathbb{R})$ on $\mathscr{P}_{n,m}$. We see that the homogeneous space (IV) is diffeomorphic to the homogeneous space $\mathscr{P}_{n,m}$. It is shown that the algebra $\mathbb{D}$(IV) of $GL_{n,m}(\mathbb{R})$-invariant differential operators on the homogeneous space (IV) is not commutative. So far, nobody has found a set of generators of $\mathbb{D}$(IV). We provide some examples of explicit invariant differential operators on $\mathscr{P}_{n,m}$ and investigate invariant differential operators on (IV). We provide some open problems that should be solved in the future. In Section 7, we study $SL_{n,m}(\mathbb{R})$-invariant differential operators on the non-symmetric homogeneous space (V). The semidirect product $SL_{n,m}(\mathbb{R})$ of $SL(n,\mathbb{R})$ and the additive group $\mathbb{R}^{(m,n)}$ given by

$$SL_{n,m}(\mathbb{R}) := SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

acts naturally and transitively on the following space:

$$\mathfrak{P}_{n,m} := \mathfrak{P}_n \times \mathbb{R}^{(m,n)} = \{\, Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0,\ \det(Y) = 1 \,\} \times \mathbb{R}^{(m,n)}. \tag{6}$$

See Formula (81) for the precise action of $SL_{n,m}(\mathbb{R})$ on $\mathfrak{P}_{n,m}$. We see that the homogeneous space (V) is diffeomorphic to the homogeneous space $\mathfrak{P}_{n,m}$. It is shown that the algebra $\mathbb{D}$(V) of $SL_{n,m}(\mathbb{R})$-invariant differential operators on the homogeneous space (V) is not commutative. So far, nobody has found a set of generators of $\mathbb{D}$(V). We provide some examples of explicit invariant differential operators on $\mathfrak{P}_{n,m}$ and investigate invariant differential operators on (V). We provide some open problems that should be solved in the future. In the final section, we study G^J-invariant differential operators on the non-symmetric homogeneous space (VI). The homogeneous space (VI) is biholomorphic to the so-called Siegel–Jacobi space $\mathbb{H}_{n,m}$ given by

$$\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)} = \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im}\Omega > 0 \,\} \times \mathbb{C}^{(m,n)}. \tag{7}$$

The Jacobi group

$$G^J := Sp(2n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \qquad \text{(semidirect product)}$$

acts naturally and transitively on the Siegel–Jacobi space $\mathbb{H}_{n,m}$. See Formula (96) for the action of G^J on $\mathbb{H}_{n,m}$. Here,

$$H_{\mathbb{R}}^{(n,m)} = \{(\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \ \text{symmetric}\,\} \tag{8}$$

denotes the Heisenberg group endowed with the following multiplication:

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') = (\lambda+\lambda', \mu+\mu'; \kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda')$$

with $(\lambda,\mu;\kappa)$, $(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$. It is shown that the algebra $\mathbb{D}$(VI) of G^J-invariant differential operators on the homogeneous space (VI) is not commutative. So far, nobody has found a set of generators of $\mathbb{D}$(VI). We provide some examples of explicit invariant differential operators on $\mathbb{H}_{n,m}$ and investigate invariant differential operators on (VI). We provide some open problems that should be solved in the future. Using the commutative subalgebra of $\mathbb{D}$(VI) containing the Laplace operator of $\mathbb{H}_{n,m}$, we introduce the notion of Maass–Jacobi functions.

Notations. We denote by $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers, respectively. $\mathbb{R}^\times$ (resp. $\mathbb{C}^\times$) denotes the group of nonzero real (resp. complex) numbers. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. For a square matrix $A\in F^{(k,k)}$ of degree k, $\mathrm{Tr}(A)$ denotes the trace of A. For any $M\in F^{(k,l)}$, ${}^tM$ denotes the transpose of M. For a positive integer n, I_n denotes the identity matrix of degree n. For $A\in F^{(k,l)}$ and $B\in F^{(k,k)}$, we set $B[A]={}^tABA$ (Siegel's notation). For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. $\mathrm{diag}(a_1,\cdots,a_n)$ denotes the $n\times n$ diagonal matrix with diagonal entries $a_1,\cdots,a_n$. For a square matrix Ω, $\mathrm{Im}\,\Omega$ denotes the imaginary part of Ω. For a smooth manifold X, we denote by $C^\infty(X)$ (resp. $C_c^\infty(X)$) the algebra of all infinitely differentiable functions (resp. with compact support) on X. $O(n):=O(n,\mathbb{R})=\{\,g\in GL(n,\mathbb{R})\mid g\,{}^tg={}^tgg=I_n\,\}$ is the real orthogonal matrix of degree n. $SO(n):=SO(n,\mathbb{R})=O(n)\cap SL(n,\mathbb{R})$.

We denote

$$\begin{aligned}
GL_{n,m}(\mathbb{R})&=GL(n,\mathbb{R})\ltimes\mathbb{R}^{(m,n)},\quad GL_{n,m}(\mathbb{Z})=GL(n,\mathbb{Z})\ltimes\mathbb{Z}^{(m,n)},\\
SL_{n,m}(\mathbb{R})&=SL(n,\mathbb{R})\ltimes\mathbb{R}^{(m,n)},\quad SL_{n,m}(\mathbb{Z})=SL(n,\mathbb{Z})\ltimes\mathbb{Z}^{(m,n)},\\
\mathscr{P}_n&=\{Y\in\mathbb{R}^{(n,n)}\mid Y={}^tY>0\,\}\cong GL(n,\mathbb{R})/O(n,\mathbb{R}),\\
\mathfrak{P}_n&=\{Y\in\mathbb{R}^{(n,n)}\mid Y={}^tY>0,\ \det(Y)=1\,\}\cong SL(n,\mathbb{R})/SO(n,\mathbb{R}),\\
&\mathfrak{H}_n\ (\text{see Definition 4}),\\
\mathscr{P}_{n,m}&=\mathscr{P}_n\times\mathbb{R}^{(m,n)}\cong GL_{n,m}(\mathbb{R})/O(n,\mathbb{R}),\\
\mathfrak{P}_{n,m}&=\mathfrak{P}_n\times\mathbb{R}^{(m,n)}\cong SL_{n,m}(\mathbb{R})/SO(n,\mathbb{R}),\\
\mathfrak{H}_{n,m}&=\mathfrak{H}_n\times\mathbb{R}^{(m,n)}\cong SL_{n,m}(\mathbb{R})/SO(n,\mathbb{R}),\\
\Gamma_n&=GL(n,\mathbb{Z}),\quad \Gamma^n=SL(n,\mathbb{Z}),\\
\mathfrak{R}_n&=GL(n,\mathbb{Z})\backslash\mathscr{P}_n\cong GL(n,\mathbb{Z})\backslash GL(n,\mathbb{R})/O(n,\mathbb{R}),\\
\mathfrak{S}_n&=SL(n,\mathbb{Z})\backslash\mathfrak{P}_n\cong SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n,\mathbb{R}),\\
\mathfrak{R}_{n,m}&=GL_{n,m}(\mathbb{Z})\backslash\mathscr{P}_{n,m}\cong GL_{n,m}(\mathbb{Z})\backslash GL_{n,m}(\mathbb{R})/O(n,\mathbb{R}),\\
\mathfrak{S}_{n,m}&=SL_{n,m}(\mathbb{Z})\backslash\mathfrak{P}_{n,m}\cong SL_{n,m}(\mathbb{Z})\backslash SL_{n,m}(\mathbb{R})/SO(n,\mathbb{R}).
\end{aligned}$$

Here, " $\cong$ " denotes the diffeomorphism.

$$J_n=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$$

denotes the symplectic matrix of degree $2n$.

$$\mathbb{H}_n=\{\,\Omega\in\mathbb{C}^{(n,n)}\mid\Omega={}^t\Omega,\quad \mathrm{Im}\,\Omega>0\,\}$$

denotes the Siegel upper half plane of degree n.

$$Sp(2n,\mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n \}$$

denotes the symplectic group of degree n and

$$\Gamma_n^\flat = Sp(2n,\mathbb{Z}) = \{ \gamma \in \mathbb{Z}^{(2n,2n)} \mid {}^t\gamma J_n\gamma = J_n \} \subset Sp(2n,\mathbb{R})$$

denotes the Siegel modular group of degree n. We let

$$G^J = Sp(2n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

be the Jacobi group. Here, $H_{\mathbb{R}}^{(n,m)}$ is the Heisenberg group (see Formula (8) for the precise definition). We put $\Gamma^J := \Gamma_n^\flat \ltimes H_{\mathbb{Z}}^{(n,m)}$.

$$\mathfrak{X}_n := \Gamma_n^\flat \backslash \mathbb{H}_n = Sp(2n,\mathbb{Z}) \backslash Sp(2n,\mathbb{R})/U(n)$$

and

$$\mathfrak{X}_{n,m} := \Gamma^J \backslash \mathbb{H}_{n,m} = (Sp(2n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}) \backslash G^J/(U(n) \times S(m,\mathbb{R})).$$

$\mathbb{D}(\mathscr{P}_n)$ denotes the algebra of all $GL(n,\mathbb{R})$-invariant differential operators on $\mathscr{P}_n$. $\mathbb{D}(\mathfrak{P}_n)$ denotes the algebra of all $SL(n,\mathbb{R})$-invariant differential operators on $\mathfrak{P}_n$. $\mathbb{D}(\mathfrak{H}_n)$ denotes the algebra of all $SL(n,\mathbb{R})$-invariant differential operators on $\mathfrak{H}_n$. $\mathbb{D}(\mathbb{H}_n)$ denotes the algebra of all $Sp(2n,\mathbb{R})$-invariant differential operators on $\mathbb{H}_n$. $\mathbb{D}(\mathscr{P}_{n,m})$ denotes the algebra of all $GL_{n,m}(\mathbb{R})$-invariant differential operators on $\mathscr{P}_{n,m}$. $\mathbb{D}(\mathfrak{P}_{n,m})$ denotes the algebra of all $SL_{n,m}(\mathbb{R})$-invariant differential operators on $\mathfrak{P}_{n,m}$. $\mathbb{D}(\mathfrak{H}_{n,m})$ denotes the algebra of all $SL_{n,m}(\mathbb{R})$-invariant differential operators on $\mathfrak{H}_{n,m}$. $\mathbb{D}(\mathbb{H}_{n,m})$ denotes the algebra of all G^J-invariant differential operators on $\mathbb{H}_{n,m}$. $\mathscr{Z}_{n,m}$ denotes the center of $\mathbb{D}(\mathscr{P}_{n,m})$. $\mathfrak{Z}_{n,m}$ denotes the center of $\mathbb{D}(\mathfrak{P}_{n,m})$. $\widetilde{\mathfrak{Z}}_{n,m}$ denotes the center of $\mathbb{D}(\mathfrak{H}_{n,m})$. $\mathfrak{C}_{n,m}$ denotes the center of $\mathbb{D}(\mathbb{H}_{n,m})$.

2. Preliminaries

Throughout this section, we let G be a connected real Lie group of finite dimension n and let K be a subgroup of G. We let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). The symmetric algebra $S(\mathfrak{g})$ is defined to be the algebra of complex-valued polynomial functions on the dual space $\mathfrak{g}^*$. If $X_1, \cdots, X_n$ is a basis of $\mathfrak{g}$, $S(\mathfrak{g})$ can be identified with the algebra of all polynomials

$$\sum_{(k)} a_{k_1\cdots k_n} X_1^{k_1} \cdots X_n^{k_n}, \quad (k) = (k_1, \cdots, k_n) \in (\mathbb{Z}_+)^n.$$

Here, $\mathbb{Z}_+$ denotes the set of all non-negative integers. For an element $g \in G$, L_g (resp. R_g) is the left (resp. right) translation by g defined by

$$L_g(h) = gh \text{ (resp. } R_g(h) = hg) \qquad \text{for all } h \in G.$$

If $X \in \mathfrak{g}$, we let $\widetilde{X}$ denote a differential operator on G defined by

$$(\widetilde{X}f)(g) := \frac{\partial}{\partial t}\Big|_{t=0} f(g \cdot \exp(tX)), \qquad f \in C^\infty(G),\ g \in G.$$

$\mathbb{E}(G)$ denotes the algebra of all differential operators on G. A differential operator $D \in \mathbb{E}(G)$ is said to be left-invariant if

$$D(f \circ L_g) = D(f) \circ L_g \qquad \text{for all } f \in C^\infty(G) \text{ and } g \in G.$$

We let $\mathbb{D}(G)$ be the algebra of all left-invariant differential operators on G and let $\mathbf{Z}(G)$ be the center of $\mathbb{D}(G)$. It is easily seen that

$$\widetilde{\mathrm{ad}(X)Y} = \widetilde{X}\widetilde{Y} - \widetilde{Y}\widetilde{X} \qquad \text{for all } X, Y \in \mathfrak{g}.$$

Now, for any $X \in \mathfrak{g}$, we define the map $\mathrm{ad}(X) : \mathbb{D}(G) \longrightarrow \mathbb{D}(G)$ by

$$\mathrm{ad}(X)D := \widetilde{X}D - D\widetilde{X} \qquad \text{for all } D \in \mathbb{D}(G). \tag{9}$$

Obviously $\mathrm{ad}(X)$ is a derivation of the algebra $\mathbb{D}(G)$. We define

$$e^{\mathrm{ad}(X)}(D) := \sum_{k=0}^{\infty} \frac{1}{k!} \,(\mathrm{ad}(X))^k(D), \qquad D \in \mathbb{D}(G). \tag{10}$$

Definition 1. *The coset space G/K is said to be* reductive *if there exists a subspace $\mathfrak{p}$ such that*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \qquad \text{and} \qquad \mathrm{Ad}_G(k)\mathfrak{p} \subset \mathfrak{p} \quad \text{for all } k \in K. \tag{11}$$

Here $\mathrm{Ad}_G : G \longrightarrow GL(\mathfrak{g})$ denotes the adjoint representation of G on $\mathfrak{g}$.

Theorem 1. *We assume the coset space G/K is reductive. Then there exists a unique linear bijection (called the* symmetrization*)*

$$\lambda : S(\mathfrak{g}) \longrightarrow \mathbb{D}(G) \tag{12}$$

such that $\lambda(X^m) = \widetilde{X}^m$ ($X \in \mathfrak{g}$, $m \in \mathbb{Z}^+$). If $X_1, \cdots, X_n$ is any basis of $\mathfrak{g}$ and $P \in S(\mathfrak{g})$; then, for any $f \in C^\infty(G)$,

$$(\lambda(P)f)(g) = \left[P(\partial_1, \cdots, \partial_n) f(g \cdot \exp(t_1 X_1 + \cdots + t_n X_n)\right]_{t=0}, \tag{13}$$

where $\partial_i = \partial/\partial t_i$ ($1 \le i \le n$) and $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$. Here, the suffix $[\ \cdot\ ]_{t=0}$ means the evaluation at $t = 0$ after differentiation.

Proof. The proof can be found in [4] (Theorem 4.3, pp. 280–281). □

Definition 2. *We fix an element $g \in G$. The mapping*

$$\mathrm{Ad}(g) : \mathbb{D}(G) \longrightarrow \mathbb{D}(G)$$

is defined by

$$\mathrm{Ad}\left(g^{-1}\right)D = D^{R_g} \qquad \text{for all } D \in \mathbb{D}(G). \tag{14}$$

Here, $D^{R_g} : C^\infty(G) \longrightarrow C^\infty(G)$ is a differential operator on G defined by

$$D^{R_g} f = D(f \circ R_{g^{-1}}) \circ R_{g^{-1}} \qquad \text{for all } f \in C^\infty(G).$$

We let

$$\widetilde{\mathfrak{g}} := \{\, \widetilde{X} \in \mathbb{E}(G) \mid X \in \mathfrak{g} \,\}.$$

Since $\widetilde{\mathfrak{g}}$ generates $\mathbb{D}(G)$, we have

$$\operatorname{Ad}(\exp X)D = e^{\operatorname{ad}(X)}(D) \qquad \text{for all } X \in \mathfrak{g} \text{ and } D \in \mathbb{D}(G). \tag{15}$$

We let $I(\mathfrak{g})$ be the space of all $\operatorname{Ad}_G(G)$-invariants in $S(\mathfrak{g})$, i.e.,

$$I(\mathfrak{g}) = \{P \in S(\mathfrak{g}) \mid \operatorname{Ad}_G(g)P = P \quad \text{for all } g \in G\}.$$

Theorem 2. *We assume the coset space G/K is reductive. Then*

$$\lambda(I(\mathfrak{g})) = \mathbf{Z}(G). \tag{16}$$

Moreover, $\mathbf{Z}(G)$ consists of all bi-invariant differential operators on G.

Proof. The proof can be found in [4] (Corollary 4.5, pp. 283–284). □

If G/K is a reductive homogeneous manifold in the sense of Definition 1, we let $\pi : G \longrightarrow G/K$ be the projection map and we put $\widetilde{f} = f \circ \pi$ for a function f on G.

Theorem 3. *We assume the coset space G/K is reductive. We put*

$$\mathbb{D}_K(G) := \{\, D \in \mathbb{D}(G) \mid D^{R_k} = D \quad \text{for all } k \in K\,\}.$$

Then, the mapping

$$\mu : \mathbb{D}_K(G) \longrightarrow \mathbb{D}(G/K)$$

defined by

$$\widetilde{\mu(D)f} = D\widetilde{f}, \qquad f \in C^\infty(G/K)$$

is a surjective homomorphism. The kernel is given by

$$\ker \mu = \mathbb{D}_K(G) \cap \mathbb{D}(G)\mathfrak{k}$$

and hence we have the isomorphism

$$\mathbb{D}(G/K) \cong \mathbb{D}_K(G)/(\mathbb{D}_K(G) \cap \mathbb{D}(G)\mathfrak{k}).$$

Proof. The proof can be found in [4] (Theorem 4.6, pp. 285–286). □

Corollary 1. *We assume the coset space G/K is reductive. We let $I(\mathfrak{p})$ denote the set of all $\operatorname{Ad}_G(K)$-invariants in $S(\mathfrak{p})$. Then*

$$\mathbb{D}_K(G) = (\mathbb{D}_K(G) \cap \mathbb{D}(G)\mathfrak{k}) \oplus \lambda(I(\mathfrak{p})).$$

Proof. See [4], p. 286. □

Theorem 4. *We let G/K be a reductive homogeneous space. The mapping*

$$\Theta : I(\mathfrak{p}) \longrightarrow \mathbb{D}(G/K)$$

defined by

$$\Theta(P) := D_{\lambda(P)} \qquad \text{for all } P \in I(\mathfrak{p})$$

is a linear bijection. Explicitly, for any function $f \in C^\infty(G/K)$*,*

$$\left(D_{\lambda(D)} f\right)(gK) = \left[P(\partial_1, \cdots, \partial_n)\widetilde{f}(g \cdot \exp(t_1 X_1 + \cdots + t_r X_r)\right]_{t=0}, \tag{17}$$

where $\partial_i = \partial/\partial t_i \ (1 \le i \le r),\ t = (t_1, \cdots, t_r) \in \mathbb{R}^r$ *and* $\{X_1, \cdots, X_r\}$ *is a basis of* $\mathfrak{p}$*.*

Proof. The proof can be found in [4], Theorem 4.6, pp. 285–286. □

Remark 1. (1) Θ *is not multiplicative in general. In fact, we have*

$$\Theta(P_1 P_2) = \Theta(P_1)\Theta(P_2) + \Theta(Q) \qquad \text{for all } P_1, P_2 \in I(\mathfrak{p}),$$

where $Q \in I(\mathfrak{p})$ *has degree* $<$ *degree(*P_1*)* $+$ *degree(*P_2*).*

(2) *If* $P_1, \cdots, P_d$ *are generators of* $I(\mathfrak{p}_{\mathbb{C}})$*, then* $\Theta(P_1), \cdots, \Theta(P_d)$ *are generators of* $\mathbb{D}(G/K)$*. Here,* $\mathfrak{p}_{\mathbb{C}}$ *denotes the complexification of* $\mathfrak{p}$*.*

(3) *If* $I(\mathfrak{p})$ *has a finite system of generators* $P_1, \cdots, P_d$ *and we put* $D_i = \Theta(P_i)$ $(1 \le i \le d)$*, then each* $D \in \mathbb{D}(G/K)$ *can be written*

$$D = \sum_{(n)} a_{n_1 \cdots n_d} D_1^{n_1} \cdots D_d^{n_d},$$

where $(n) = (n_1, \cdots, n_d) \in \mathbb{Z}_+^d$*.*

Theorem 5. (1) *We let* (G, K) *be a symmetric pair (i.e.,* G/K *is a symmetric space),* G *semisimple, and* K *a maximal compact subgroup of* G*. Then* $\mathbb{D}(G/K)$ *is a commutative algebra. Here,* $\mathbb{D}(G/K)$ *denotes the algebra of all invariant differential operators on* G/K*.* (2) *We let* H *be a connected Lie group of finite dimension and let* $H^\diamondsuit$ *be the diagonal in* $H \times H$*. Under the bijection*

$$(h_1, h_2)H^\diamondsuit \mapsto h_1 h_2^{-1}, \qquad h_1, h_2 \in H$$

of $(H \times H)/H^\diamondsuit$ *onto* H*, we have the identification*

$$\mathbb{D}((H \times H)/H^\diamondsuit) = \mathbf{Z}(H).$$

Here, $\mathbb{D}((H \times H)/H^\diamondsuit)$ *(resp.* $\mathbb{D}(H)$*) denotes the algebra of invariant differential operators on* $(H \times H)/H^\diamondsuit$ *(resp.* H*) and* $\mathbf{Z}(H)$ *denotes the center of* $\mathbb{D}(H)$*.*

Proof. The proof can be found in [4] (Theorem 5.7, pp. 294–295). □

We let $M = G/K$ be a symmetric space of the noncompact type, i.e., G is a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. We let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. We let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{a}^+ \subset \mathfrak{a}$ be a fixed Weyl chamber. We let $G = KAN$ be an Iwasawa decomposition of G. We denote by $\mathbb{D}(A)$ the algebra of invariant differential operators on $A \cdot o$. Here, $o = e \cdot K$ is the origin of G/K (e is the identity element of G) and $A \cdot o := \{a \cdot o \mid a \in A\}$ denotes the A-orbit of o in G/K. We let W be the Weyl group of G, that is, the Weyl group of the root system of G.

We recall the linear bijection $\lambda : S(\mathfrak{g}) \longrightarrow \mathbb{D}(G)$ in Theorem 1. We see that $S(\mathfrak{a})$ can be identified with $\mathbb{D}(A)$. We let $\mathbb{D}_W(A)$ be the set of all W-invariant differential operators on the orbit $A \cdot o$ and $I(\mathfrak{a})$ the set of all W-invariants in $S(\mathfrak{a})$. Then, there exists a bijection of

$\mathbb{D}(G/K)$ onto $\mathbb{D}_W(A)$ (cf. see [4] (Theorem 5.13, pp. 300–302)). Furthermore, there exists a surjective homomorphism of $\mathbb{D}_K(G)$ onto $\mathbb{D}_W(A)$ with kernel

$$\mathbb{D}_K(G) \cap \mathbb{D}(G)\mathfrak{k}.$$

We refer to [4] (Theorem 5.18, p. 306) for more detail. Combining all these results, we conclude that if G/K is a symmetric space of the noncompact type, then $\mathbb{D}(G/K)$ is a polynomial algebra in r algebraically independent generators $\delta_1, \cdots, \delta_r$ whose degrees $d_1, \cdots, d_r$ are canonically determined by G. We note that $r = \dim A$ is the rank of G or the rank of G/K.

3. Invariant Differential Operators on $GL(n,\mathbb{R})/O(n,\mathbb{R})$

For any positive integer $n \geq 1$, we let

$$\mathscr{P}_n := \{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \}$$

be the open convex cone in the Euclidean space $\mathbb{R}^N$ with $N = \frac{n(n+1)}{2}$. Then, $GL(n,\mathbb{R})$ acts $\mathscr{P}_n$ transitively by

$$g \cdot Y = gY\,{}^tg, \qquad \text{where } g \in GL(n,\mathbb{R}) \text{ and } Y \in \mathscr{P}_n. \tag{18}$$

Since $O(n)$ is the isotopic subgroup of $GL(n,\mathbb{R})$ at I_n, the symmetric space $GL(n,\mathbb{R})/O(n)$ is diffeomorphoc to $\mathscr{P}_n$.

For $Y = (y_{ij}) \in \mathscr{P}_n$, we put

$$dY = (dy_{ij}) \qquad \text{and} \qquad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right).$$

For a fixed element $A \in GL(n,\mathbb{R})$, we put

$$Y_* = A \cdot Y = AY\,{}^tA, \quad Y \in \mathscr{P}_n.$$

Then

$$dY_* = A\,dY\,{}^tA \quad \text{and} \quad \frac{\partial}{\partial Y_*} = {}^tA^{-1} \frac{\partial}{\partial Y} A^{-1}. \tag{19}$$

We can see easily that for any positive real number $C > 0$,

$$ds^2_{n;C} = C \cdot \mathrm{Tr}\,((Y^{-1}dY)^2)$$

is a Riemannian metric on $\mathscr{P}_n$ invariant under Action (18) and its Laplace operator is given by

$$\Delta_{n;C} = \frac{1}{C} \cdot \mathrm{Tr}\left(\left(Y \frac{\partial}{\partial Y} \right)^2 \right),$$

where $\mathrm{Tr}\,(M)$ denotes the trace of a square matrix M. We also can see that

$$d\mu_n(Y) = (\det Y)^{-\frac{n+1}{2}} \prod_{i \leq j} dy_{ij} \tag{20}$$

is a $GL(n,\mathbb{R})$-invariant volume element on $\mathscr{P}_n$.

Theorem 6. *A geodesic $\alpha(t)$ joining I_n and $Y \in \mathscr{P}_n$ has the form*

$$\alpha(t) = \exp(tA[V]), \qquad t \in [0,1],$$

where

$$Y = (\exp A)[V] = \exp(A[V]) = \exp({}^t VAV)$$

is the spectral decomposition of Y, where $V \in O(n,\mathbb{R})$, $A = \mathrm{diag}(a_1,\cdots,a_n)$ with all $a_j \in \mathbb{R}$. The distance of $\alpha(t)$ $(0 \le t \le 1)$ between I_n and Y is

$$\left(\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}}.$$

Proof. The proof can be found in [6] (pp. 16–17). □

We consider the following Maass–Selberg (differential) operators $\delta_1, \delta_2, \cdots, \delta_n$ on $\mathscr{P}_n$ defined by

$$\delta_k = \mathrm{Tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^k\right), \quad k = 1,2,\cdots,n, \tag{21}$$

By Formula (19), we obtain

$$\left(Y_*\frac{\partial}{\partial Y_*}\right)^i = A\left(Y\frac{\partial}{\partial Y}\right)^i A^{-1}$$

for any $A \in GL(n,\mathbb{R})$, so each δ_i $(1 \le i \le n)$ is invariant under Action (18) of $GL(n,\mathbb{R})$.

Maass [1,2] and Selberg [3] proved the following.

Theorem 7. *The algebra $\mathbb{D}(\mathscr{P}_n)$ of all $GL(n,\mathbb{R})$-invariant differential operators on $\mathscr{P}_n$ is generated by $\delta_1, \delta_2, \cdots, \delta_n$. Furthermore, $\delta_1, \delta_2, \cdots, \delta_n$ are algebraically independent and $\mathbb{D}(\mathscr{P}_n)$ is isomorphic to the commutative ring $\mathbb{C}[x_1, x_2, \cdots, x_n]$ with n indeterminates $x_1, x_2, \cdots, x_n$.*

Proof. The proof can be found in [2] (pp. 64–66) and [6] (pp. 29–30). The last statement follows immediately from the work of Harish–Chandra [10,11] or [4] (p. 294). □

Remark 2. *A different description of $\mathbb{D}(\mathscr{P}_n)$ was given by Helgason [4] (Chapter II, Exercise C.1, p. 337; Solution pp. 571–572). See also [4] (Chapter II, Exercise C.8, pp. 339–340) for a related topic.*

We let $\mathfrak{g} = \mathbb{R}^{(n,n)}$ be the Lie algebra of $GL(n,\mathbb{R})$. The adjoint representation Ad of $GL(n,\mathbb{R})$ is given by

$$\mathrm{Ad}(g) = gXg^{-1}, \quad g \in GL(n,\mathbb{R}),\ X \in \mathfrak{g}.$$

The Killing form B of $\mathfrak{g}$ is given by

$$B(X,Y) = 2\,n\,\mathrm{Tr}(XY) - 2\,\mathrm{Tr}(X)\,\mathrm{Tr}(Y), \quad X, Y \in \mathfrak{g}.$$

Since $B(aI_n, X) = 0$ for all $a \in \mathbb{R}$ and $X \in \mathfrak{g}$, B is degenerate. So the Lie algebra $\mathfrak{g}$ of $GL(n,\mathbb{R})$ is not semisimple.

We put $K = O(n)$. The Lie algebra $\mathfrak{k}$ of K is given by

$$\mathfrak{k} = \left\{ X \in \mathfrak{g} \mid X + {}^tX = 0 \right\}.$$

We let $\mathfrak{p}$ be the subspace of $\mathfrak{g}$ defined by

$$\mathfrak{p} = \left\{ X \in \mathfrak{g} \mid X = {}^tX \in \mathbb{R}^{(n,n)} \right\}.$$

Then
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is the direct sum of $\mathfrak{k}$ and $\mathfrak{p}$ with respect to the Killing form B. Since $\mathrm{Ad}(k)\mathfrak{p} \subset \mathfrak{p}$ for any $k \in K,\ K$ acts on $\mathfrak{p}$ via the adjoint representation by

$$k \cdot X = \mathrm{Ad}(k)X = kX\,{}^t k, \quad k \in K,\ X \in \mathfrak{p}. \tag{22}$$

Action (22) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p})$ of $\mathfrak{p}$ and the symmetric algebra $S(\mathfrak{p})$. We denote by $\mathrm{Pol}(\mathfrak{p})^K$ (resp. $S(\mathfrak{p})^K$) the subalgebra of $\mathrm{Pol}(\mathfrak{p})$ (resp. $S(\mathfrak{p})$) consisting of all K-invariants. The following inner product $(\,,\,)$ on $\mathfrak{p}$ defined by

$$(X, Y) = B(X, Y), \quad X, Y \in \mathfrak{p}$$

gives an isomorphism as vector spaces

$$\mathfrak{p} \cong \mathfrak{p}^*, \quad X \mapsto f_X, \quad X \in \mathfrak{p}, \tag{23}$$

where $\mathfrak{p}^*$ denotes the dual space of $\mathfrak{p}$ and f_X is the linear functional on $\mathfrak{p}$ defined by

$$f_X(Y) = (Y, X), \quad Y \in \mathfrak{p}.$$

It is known that there is a canonical linear bijection of $S(\mathfrak{p})^K$ onto $\mathbb{D}(\mathscr{P}_n)$. Identifying $\mathfrak{p}$ with $\mathfrak{p}^*$ by the above isomorphism (23), we obtain a canonical linear bijection

$$\Phi_n : \mathrm{Pol}(\mathfrak{p})^K \longrightarrow \mathbb{D}(\mathscr{P}_n) \tag{24}$$

of $\mathrm{Pol}(\mathfrak{p})^K$ onto $\mathbb{D}(\mathscr{P}_n)$. The map Φ_n is described explicitly as follows. We put $N = n(n+1)/2$. We let $\{\xi_\alpha \mid 1 \le \alpha \le N\}$ be a basis of $\mathfrak{p}$. If $P \in \mathrm{Pol}(\mathfrak{p})^K$, then

$$\Big(\Phi_n(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\,\exp\left(\sum_{\alpha=1}^{N} t_\alpha \xi_\alpha\right)K\right)\right]_{(t_\alpha)=0}, \tag{25}$$

where $f \in C^\infty(\mathscr{P}_n)$. We refer to [4] (Theorem 4.6, pp. 285–286) or Formula (17) for more detail. In general, it is very hard to express $\Phi_n(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p})^K$.

We let
$$q_i(X) = \mathrm{Tr}(X^i), \quad i = 1, 2, \cdots, n \tag{26}$$

be the polynomials on $\mathfrak{p}$. Here, we take a coordinate $x_{11}, x_{12}, \cdots, x_{nn}$ in $\mathfrak{p}$ given by

$$X = \begin{pmatrix} x_{11} & \frac{1}{2}x_{12} & \cdots & \frac{1}{2}x_{1n} \\ \frac{1}{2}x_{12} & x_{22} & \cdots & \frac{1}{2}x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}x_{1n} & \frac{1}{2}x_{2n} & \cdots & x_{nn} \end{pmatrix}.$$

For any $k \in K$,

$$(k \cdot q_i)(X) = q_i(k^{-1}Xk) = \mathrm{Tr}(k^{-1}X^i k) = q_i(X), \quad i = 1, 2, \cdots, n.$$

Thus, $q_i \in \mathrm{Pol}(\mathfrak{p})^K$ for $i = 1, 2, \cdots, n$. By a classical invariant theory (cf. [12,13]), we can prove that the algebra $\mathrm{Pol}(\mathfrak{p})^K$ is generated by the polynomials $q_1, q_2, \cdots, q_n$ and that $q_1, q_2, \cdots, q_n$ are algebraically independent. Using Formula (26), we can show without difficulty that

$$\Phi_n(q_1) = \mathrm{Tr}\left(2Y \frac{\partial}{\partial Y}\right).$$

However, $\Phi_n(q_i)$ $(i = 2, 3, \cdots, n)$ are still not known explicitly.

We propose the following conjecture.

Conjecture 8. *For any n,*

$$\Phi_n(q_i) = \mathrm{Tr}\left(\left(2Y\frac{\partial}{\partial Y}\right)^i\right), \quad i = 1, 2, \cdots, n.$$

Remark 3. *The above conjecture is true for $n = 1, 2$.*

The fundamental domain $\mathfrak{M}_n$ for $GL(n, \mathbb{Z})$ in $\mathscr{P}_n$ which was found by H. Minkowski [14] is defined as a subset of $\mathscr{P}_n$ consisting of $Y = (y_{ij}) \in \mathscr{P}_n$ satisfying the following conditions (M.1)–(M.2) (cf. [2], (p. 123)):

(M.1) $aY\,{}^t a \geq y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^n$ in which $a_k, \cdots, a_n$ are relatively prime for $k = 1, 2, \cdots, n$.

(M.2) $y_{k,k+1} \geq 0$ for $k = 1, \cdots, n-1$.

We say that a point of $\mathfrak{M}_n$ is *Minkowski reduced* or simply *M-reduced*. $\mathfrak{M}_n$ is a convex cone through the origin bounded by a finite number of hyperplanes and is closed in $\mathscr{P}_n$ (cf. [2], pp. 123–124). Thus, we see that $\mathfrak{M}_n$ is a semi-algebraic set with real analytic structure.

We let

$$\mathfrak{R}_n := GL(n, \mathbb{Z})\backslash GL(n, \mathbb{R})/O(n, \mathbb{R}) = GL(n, \mathbb{Z})\backslash \mathscr{P}_n$$

be the locally symmetric space. $\mathscr{P}_n$ parameterizes principally polarized real tori of dimension n (cf. [15]). The arithmetic quotient $\mathfrak{R}_n$ is the moduli space of isomorphism classes of principally polarized real tori of dimension n. Unfortunately $\mathfrak{R}_n$ does not admit the structure of a real algebraic variety and does not admit a compactification which is defined over the rational number field $\mathbb{Q}$ (cf. [16] or [17]).

We offer the definition of automorphic forms for $GL(n, \mathbb{Z})$ given by A. Terras (cf. [6], (p. 182)).

For $s = (s_1, \cdots, s_n) \in \mathbb{C}^n$, Atle Selberg [3] (pp. 57–58) introduced the power function $p_s : \mathscr{P}_n \longrightarrow \mathbb{C}$ defined by

$$p_s(Y) := \prod_{j=1}^{n} (\det Y_j)^{s_j}, \quad Y \in \mathscr{P}_n, \tag{27}$$

where $Y_j \in \mathscr{P}_j$ $(1 \leq j \leq n)$ is the $j \times j$ upper left corner of Y. It is known that $p_s(Y)$ is a joint eigenfunction of $\mathbb{D}(\mathscr{P}_n)$, i.e., $p_s(Y)$ is an eigenfunction of each invariant differential operator in $\mathbb{D}(\mathscr{P}_n)$ (cf. [6], (pp. 39–40)).

Definition 3. *A real analytic function $f : \mathscr{P}_n \longrightarrow \mathbb{C}$ is said to be a* automorphic form *for $GL(n, \mathbb{Z})$ if it satisfies the following Conditions* (A1)–(A3)*:*

(A1) $f(\gamma Y\,{}^t\gamma) = f(Y)$ *for all $Y \in \mathscr{P}_n$ and all $\gamma \in GL(n, \mathbb{Z})$;*

(A2) *f is an eigenfunction of all $D \in \mathbb{D}(\mathscr{P}_n)$, i.e., $Df = \lambda_D f$ for some eigenvalue λ_D;*

(A3) *f has at most polynomial growth at infinity, i.e.,*

$$|f(Y)| \leq C\,|p_s(Y)| \qquad \text{for some } s \in \mathbb{C}^n \text{ and } C > 0.$$

We set $\Gamma_n = GL(n,\mathbb{Z})$. We denote by $\mathbf{A}(\Gamma_n,\lambda)$ the space of all automorphic forms for Γ_n with a given eigenvalue system λ. An automorphic form f in $\mathbf{A}(\Gamma_n,\lambda)$ is called a cusp form for Γ_n if for any k with $1 \le k \le n-1$. We have

$$\int_{X\in T^{(k,n-k)}} f\left({}^t\!\begin{pmatrix} I_k & x \\ 0 & I_{n-k} \end{pmatrix} Y \begin{pmatrix} I_k & x \\ 0 & I_{n-k} \end{pmatrix} \right) dx = 0 \quad \text{for all } Y \in \mathscr{P}_n. \tag{28}$$

Here, $T = \mathbb{R}/\mathbb{Z}$ denotes a circle, that is, a one-dimensional torus, and $T^{(k,n-k)}$ denotes the set of all $k \times (n-k)$ matrices with entries in T. Condition (A3) implies the vanishing of the constant terms in some Fourier expansions of $f(Y)$ as a periodic function in the x-variable in partial Iwasawa coordinates.

Remark 4. *Borel and Jacquet defined automorphic forms for a connected reductive group over* $\mathbb{Q}$ (cf. *[18], (pp. 199–200) and [19], (pp. 189–190)). The definition given by Borel and Jacquet is slightly different from Definition 3 given by Terras.*

One of the motivations to study automorphic forms for $GL(n,\mathbb{Z})$ is the need to study various kinds of L-functions with many gamma factors in their functional equations. Another motivation for the study of automorphic forms for $GL(n,\mathbb{Z})$ is to develop the theory of harmonic analysis on $L^2(GL(n,\mathbb{Z})\backslash\mathscr{P}_n)$ and $L^2(GL(n,\mathbb{Z})\backslash GL(n,\mathbb{R}))$ which involves the unitary representations of $GL(n,\mathbb{R})$.

Remark 5. *Grenier investigated a fundamental domain for* $GL(n,\mathbb{Z})$ *on* $\mathscr{P}_n$ *and constructed a compactification of* $GL(n,\mathbb{Z})\backslash\mathscr{P}_n$ (cf. *[20,21]).*

Remark 6. *Using the Grenier operator defined by Douglas Grenier* (cf. *[22]), we can define the notion of* stable automorphic forms *for* $GL(n,\mathbb{Z})$.

4. Invariant Differential Operators on $SL(n,\mathbb{R})/SO(n,\mathbb{R})$

First of all, we provide some geometric properties on $SL(n,\mathbb{R})/SO(n,\mathbb{R})$.

We let

$$\mathfrak{P}_n := \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0,\ \det(Y) = 1 \right\}$$

be a symmetric space associated to $SL(n,\mathbb{R})$. Indeed, $SL(n,\mathbb{R})$ acts on $\mathfrak{P}_n$ transitively by

$$g \circ Y = gY\,{}^tg, \qquad g \in SL(n,\mathbb{R}),\ Y \in \mathfrak{P}_n. \tag{29}$$

Thus, $\mathfrak{P}_n$ is a smooth manifold diffeomorphic to the symmetric space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$ through the bijective map

$$SL(n,\mathbb{R})/SO(n,\mathbb{R}) \longrightarrow \mathfrak{P}_n, \qquad g\cdot SO(n,\mathbb{R}) \mapsto g\circ I_n = g\,{}^tg, \quad g \in SL(n,\mathbb{R}).$$

For $Y \in \mathfrak{P}_n$, we have a partial Iwasawa decomposition

$$Y = \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W \end{pmatrix} \left[\begin{pmatrix} 1 & {}^tx \\ 0 & I_{n-1} \end{pmatrix} \right] = \begin{pmatrix} v^{-1} & v^{-1}\,{}^tx \\ v^{-1}x & v^{-1}x\,{}^tx + v^{1/(n-1)}W \end{pmatrix} \tag{30}$$

where $v > 0,\ x \in \mathbb{R}^{(n-1,1)}$ and $W \in \mathfrak{P}_{n-1}$. From now on, for brevity, we write $Y = [v,x,W]$ instead of Decomposition (30). In these coordinates, $Y = [v,x,W]$,

$$ds_Y^2 = \frac{n}{n-1}\,v^{-2}dv^2 + 2\,v^{-n/(n-1)}W^{-1}[dx] + ds_W^2$$

is a $SL(n,\mathbb{R})$-invariant metric on $\mathfrak{P}_n$, where $dx = {}^t(dx_1, \cdots, dx_{n-1})$ and ds^2_W is a $SL(n-1,\mathbb{R})$-invariant metric on $\mathfrak{P}_{n-1}$. The Laplace operator Δ_n of $(\mathfrak{P}_n, ds^2_Y)$ is given by

$$\Delta_n = \frac{n-1}{n}\, v^2 \frac{\partial^2}{\partial v^2} - \frac{1}{n}\frac{\partial}{\partial v} + v^{n/(n-1)}\, W\left[\frac{\partial}{\partial x}\right] + \Delta_{n-1}$$

inductively, where if $x = {}^t(x_1, \cdots, x_{n-1}) \in \mathbb{R}^{(n-1,1)}$,

$$\frac{\partial}{\partial x} = {}^t\!\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-1}}\right)$$

and Δ_{n-1} is the Laplace operator of $(\mathfrak{P}_{n-1}, ds^2_W)$,

$$d\nu_n = v^{-(n+2)/2}\, dv\, dx\, d\nu_{n-1}$$

is a $SL(n,\mathbb{R})$-invariant volume element on $\mathfrak{P}_n$ where $dx = dx_1 \cdots dx_{n-1}$ and $d\nu_{n-1}$ is a $SL(n-1,\mathbb{R})$-invariant volume element on $\mathfrak{P}_{n-1}$.

Following earlier work of Minkowski [14], Siegel [23] showed that the volume of the arithmetic quotient $SL(n,\mathbb{Z})\backslash\mathfrak{P}_n$ is given as follows:

$$\mathrm{Vol}(SL(n,\mathbb{Z})\backslash\mathfrak{P}_n) = \int_{SL(n,\mathbb{Z})\backslash\mathfrak{P}_n} d\nu_n = n\, 2^{n-1} \prod_{k=2}^{n} \frac{\zeta(k)}{\mathrm{Vol}(S^{k-1})}, \tag{31}$$

where

$$\mathrm{Vol}(S^{k-1}) = \frac{2\,(\sqrt{\pi})^k}{\Gamma(k/2)}$$

denotes the volume of the $(k-1)$-dimensional sphere S^{k-1}, $\Gamma(x)$ denotes the usual Gamma function, and $\zeta(k) = \sum_{m=1}^{\infty} m^{-k}$ denotes the Riemann zeta function. The proof of (31) can be found in [24] or [7] (pp. 27–37).

We let $\mathbb{D}(\mathfrak{P}_n)$ be the algebra of all differential operators on $\mathfrak{P}_n$ invariant under the action (29) of $SL(n,\mathbb{R})$. It is well known (cf. [4,10,11]) that $\mathbb{D}(\mathfrak{P}_n)$ is commutative and is isomorphic to the polynomial algebra $\mathbb{C}[x_1, x_2, \cdots, x_{n-1}]$ with n indeterminates $x_1, x_2, \cdots, x_{n-1}$. We observe that $n-1$ is the rank of $SL(n,\mathbb{R})$, i.e., the rank of the symmetric space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$.

In [5], using the Maass–Selberg operators $\delta_1, \delta_2, \cdots, \delta_n$ (see Formula (21)), Brennecken, Ciardo, and Hilgert found explicit generators $E_1, E_2, \cdots, E_{n-1}$ of $\mathbb{D}(\mathfrak{P}_n)$. Obviously, $E_1, E_2, \cdots, E_{n-1}$ are algebraically independent. We briefly sketch their method of finding generators $E_1, E_2, \cdots, E_{n-1}$ of $\mathbb{D}(\mathfrak{P}_n)$.

We denote by $\mathbb{D}(GL(n,\mathbb{R})/O(n,\mathbb{R}))$ (resp. $\mathbb{D}(SL(n,\mathbb{R})/SO(n,\mathbb{R}))$) the algebra of all $GL(n,\mathbb{R})$ (resp. $SL(n,\mathbb{R})$)-invariant differential operators on $GL(n,\mathbb{R})/O(n,\mathbb{R})$ (resp. $SL(n,\mathbb{R})/SO(n,\mathbb{R})$). Let us consider the following two mappings,

$$\phi : SL(n,\mathbb{R})/SO(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})/O(n,\mathbb{R}) \tag{32}$$

defined by

$$\phi(g \cdot SO(n,\mathbb{R})) := g \cdot O(n,\mathbb{R}) \qquad \text{for all } g \in SL(n,\mathbb{R})$$

and

$$p : GL(n,\mathbb{R})^+/SO(n,\mathbb{R}) \longrightarrow SL(n,\mathbb{R})/SO(n,\mathbb{R}) \tag{33}$$

defined by

$$p(g \cdot SO(n,\mathbb{R})) := (\det(g))^{-1/n} g \cdot SO(n,\mathbb{R}) \qquad \text{for all } g \in GL(n,\mathbb{R})^+.$$

Here,

$$GL(n,\mathbb{R})^{+} := \{\, g \in GL(n,\mathbb{R}) \mid \det(g) > 0 \,\}$$

is a subgroup of $GL(n,\mathbb{R})$. It is easily seen that the mapping

$$q : GL(n,\mathbb{R})^{+}/SO(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})/O(n,\mathbb{R}) \tag{34}$$

defined by

$$q(g \cdot SO(n,\mathbb{R})) := g \cdot O(n,\mathbb{R}) \qquad \text{for all } g \in GL(n,\mathbb{R})^{+}$$

is a diffeomorphism. We use this fact to identify $GL(n,\mathbb{R})^{+}/SO(n,\mathbb{R})$ with $GL(n,\mathbb{R})/O(n,\mathbb{R})$. Now, L_g denotes the left translation by g on $GL(n,\mathbb{R})/O(n,\mathbb{R})$ and $SL(n,\mathbb{R})/SO(n,\mathbb{R})$. Brennecken, Ciardo, and Hilgert [5] show the following properties (BCH1)–(BCH4):

(BCH1) The maps ϕ and p are $SL(n,\mathbb{R})$-equivariant, i.e.,

$$\phi \circ L_g = L_g \circ \phi \quad \text{and} \quad p \circ L_g = L_g \circ p \qquad \text{for all } g \in SL(n,\mathbb{R})$$

(BCH2) The mapping

$$\mathscr{L} : \mathbb{D}(GL(n,\mathbb{R})/O(n,\mathbb{R})) \longrightarrow \mathbb{D}(SL(n,\mathbb{R})/SO(n,\mathbb{R})) \tag{35}$$

defined by

$$\mathscr{L}(D)f := D(f \circ p) \circ \phi$$

for all $D \in \mathbb{D}(GL(n,\mathbb{R})/O(n,\mathbb{R}))$ and $f \in C^{\infty}(SL(n,\mathbb{R})/SO(n,\mathbb{R}))$ is a morphism of algebras. Here, we note that we identified $GL(n,\mathbb{R})^{+}/SO(n,\mathbb{R})$ with $GL(n,\mathbb{R})/O(n,\mathbb{R})$. Furthermore,

$$(\mathscr{L}(D)f) \circ \phi = D(f \circ p)$$

for all $D \in \mathbb{D}(GL(n,\mathbb{R})/O(n,\mathbb{R}))$ and $f \in C^{\infty}(SL(n,\mathbb{R})/SO(n,\mathbb{R}))$.

(BCH3) We let

$$S_n(\mathbb{R}) := \{\, X \in \mathbb{R}^{(n,n)} \mid X = {}^{t}X \,\}.$$

According to Theorem 4 or [4] (Theorem 4.6, pp. 285–286), for each $D \in \mathbb{D}(GL(n,\mathbb{R})/O(n,\mathbb{R}))$, there exists a polynomial Q_D on $S_n(\mathbb{R})$ such that

$$(Df)(g \cdot O(n,\mathbb{R})) = Q_D\left(\frac{\partial}{\partial X}\right)\bigg|_{X=0} f(g \cdot \exp(X) \cdot O(n,\mathbb{R})),$$

where $X = (x_{ij}) \in \mathbb{R}^{(n,n)}$ with $x_{ij} = x_{ji}$ $(1 \le i,j \le n)$ and

$$\frac{\partial}{\partial X} = \left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial x_{ij}}\right).$$

Here, δ_{ij} is the Kronecker delta symbol. They show that each $f \in C^{\infty}(SL(n,\mathbb{R})/SO(n,\mathbb{R}))$,

$$(\mathscr{L}(D)f)(g \cdot SO(n,\mathbb{R})) = Q_D\left(\frac{\partial}{\partial X}\right)\bigg|_{X=0} f\Big((g \cdot \exp(X - n^{-1}\,\mathrm{Tr}(X)\, I_n) \cdot SO(n,\mathbb{R}))\Big).$$

(BCH4) The morphism $\mathscr{L}$ is surjective and $\mathscr{L}(\delta_1) = 0$.

Combining the above properties (BCH1)–(BCH4), proved the following theorem was proven (cf. [5], (Theorem 3.5)):

Theorem 9. *We let $\delta_1, \delta_2, \cdots, \delta_n$ be the Maass–Selberg operators. Then $\mathscr{L}(\delta_1) = 0$ and $\mathscr{L}(\delta_k)$ $(2 \le k \le n)$ are given by*

$$\begin{aligned} &\mathscr{L}(\delta_k) f(g \cdot SO(n,\mathbb{R})) \\ = \;& \mathrm{Tr}\left(\left(\frac{\partial}{\partial X} \right)^k \right) \bigg|_{X=0} f\Big((g \cdot \exp(X - n^{-1}\,\mathrm{Tr}(X)\, I_n) \cdot SO(n,\mathbb{R})) \Big), \end{aligned}$$

where $X = (x_{ij}) \in \mathbb{R}^{(n,n)}$ with $x_{ij} = x_{ji}$ $(1 \le i, j \le n)$ and

$$\frac{\partial}{\partial X} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial x_{ij}} \right).$$

The differential operators $\mathscr{L}(\delta_2), \mathscr{L}(\delta_3), \cdots, \mathscr{L}(\delta_n)$ are algebraically independent generators of $\mathbb{D}(SL(n,\mathbb{R})/SO(n,\mathbb{R}))$.

Corollary 2. *We let*

$$S_{n;0}(\mathbb{R}) := \{\, Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY,\ \mathrm{Tr}\,(Y) = 0 \,\}.$$

Then, for each $f \in C^\infty(SL(n,\mathbb{R})/SO(n,\mathbb{R}))$ and $Y \in S_{n;0}(\mathbb{R})$, we have

$$\mathscr{L}(\delta_k) f(g \cdot SO(n,\mathbb{R})) = \mathrm{Tr}\left(\left(\frac{\partial}{\partial Y} \right)^k \right) \bigg|_{Y=0} f(g \cdot \exp(Y)) \cdot SO(n,\mathbb{R}), \quad 2 \le k \le n.$$

We let $\mathfrak{g}$ be the Lie algebra of $SL(n,\mathbb{R})$. The adjoint representation Ad of $SL(n,\mathbb{R})$ is given by

$$\mathrm{Ad}(g) = gXg^{-1}, \quad g \in SL(n,\mathbb{R}),\ X \in \mathfrak{g}.$$

The Killing form B of $\mathfrak{g}$ is given by

$$B(X,Y) = 2\,n\,\mathrm{Tr}(XY), \quad X, Y \in \mathfrak{g}.$$

For brevity, we put $K = SO(n,\mathbb{R})$. The Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k} = \left\{ X \in \mathfrak{g} \mid X + {}^tX = 0 \right\}.$$

We let $\mathfrak{p}_0$ be the subspace of $\mathfrak{g}$ defined by

$$\mathfrak{p}_0 = \Big\{ X \in \mathfrak{g} \mid X = {}^tX \in \mathbb{R}^{(n,n)}, \quad \mathrm{Tr}(X) = 0 \Big\}.$$

Then,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_0$$

is the direct sum of $\mathfrak{k}$ and $\mathfrak{p}_0$ with respect to the Killing form B, since $\mathrm{Ad}(k)\mathfrak{p}_0 \subset \mathfrak{p}_0$ for any $k \in K$, K acts on $\mathfrak{p}_0$ via the adjoint representation by

$$k \cdot X = \mathrm{Ad}(k)X = kX\,{}^tk, \quad k \in K,\ X \in \mathfrak{p}_0. \tag{36}$$

Action (36) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p}_0)$ of $\mathfrak{p}_0$ and the symmetric algebra $S(\mathfrak{p}_0)$. We denote by $\mathrm{Pol}(\mathfrak{p}_0)^K$ (resp. $S(\mathfrak{p}_0)^K$) the subalgebra of

$\mathrm{Pol}(\mathfrak{p}_0)$ (resp. $S(\mathfrak{p}_0)$) consisting of all K-invariants. The following inner product $(\,,\,)_0$ on $\mathfrak{p}_0$ defined by

$$(X,Y)_0 = B(X,Y), \quad X,Y \in \mathfrak{p}_0$$

gives an isomorphism as vector spaces

$$\mathfrak{p}_0 \cong \mathfrak{p}_0^*, \quad X \mapsto g_X, \quad X \in \mathfrak{p}_0, \tag{37}$$

where $\mathfrak{p}_0^*$ denotes the dual space of $\mathfrak{p}_0$ and g_X is the linear functional on $\mathfrak{p}_0$ defined by

$$g_X(Y) = (Y,X)_0, \quad Y \in \mathfrak{p}_0.$$

It is known that there is a canonical linear bijection of $S(\mathfrak{p}_0)^K$ onto $\mathbb{D}(\mathfrak{P})$. Identifying $\mathfrak{p}_0$ with $\mathfrak{p}_0^*$ by the above isomorphism (37), we obtain a canonical linear bijection

$$\Psi_n : \mathrm{Pol}(\mathfrak{p}_0)^K \longrightarrow \mathbb{D}(\mathfrak{P}_n) \tag{38}$$

of $\mathrm{Pol}(\mathfrak{p}_0)^K$ onto $\mathbb{D}(\mathfrak{P}_n)$. The map Ψ_n is described explicitly as follows. We put $N_0 = n(n+1)/2 - 1$. We let $\{\xi_\alpha \mid 1 \le \alpha \le N_0\}$ be a basis of $\mathfrak{p}_0$. If $P \in \mathrm{Pol}(\mathfrak{p}_0)^K$, then

$$\Big(\Psi_n(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \cdot \exp\left(\sum_{\alpha=1}^{N_0} t_\alpha \xi_\alpha\right)K\right)\right]_{(t_\alpha)=0}, \tag{39}$$

where $f \in C^\infty(\mathfrak{P}_n)$. We refer to [4] (Theorem 4.6, pp. 285–286) or Formula (17) for more detail. In general, it is very hard to express $\Psi_n(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p}_0)^K$.

If we repeat a partial decomposition process for $Y \in \mathfrak{P}_n$, we obtain the Iwasawa decomposition

$$Y = y^{-1}\mathrm{diag}\Big(1, y_1^2, (y_1y_2)^2, \cdots, (y_1y_2\cdots y_{n-1})^2\Big)\left[\begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ 0 & 1 & \cdots & x_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}\right], \tag{40}$$

where $y > 0$, $y_j \in \mathbb{R}\,(1 \le j \le n-1)$ and $x_{ij} \in \mathbb{R}\,(1 \le i < j \le n)$. Here, $y = y_1^{2(n-1)} \cdots y_{n-1}^2$ and $\mathrm{diag}(a_1, \cdots, a_n)$ denotes the $n \times n$ diagonal matrix with diagonal entries $a_1, \cdots, a_n$. In this case, we denote $Y = (y_1, \cdots, y_{n-1}, x_{12}, \cdots, x_{n-1,n})$.

We define $\Gamma_n = GL(n,\mathbb{Z})/\{\pm I_n\}$. We observe that $\Gamma_n = SL(n,\mathbb{Z})/\{\pm I_n\}$ if n is even and $\Gamma_n = SL(n,\mathbb{Z})$ if n is odd. An automorphic form for Γ_n is defined to be a real analytic function $f : \mathfrak{P}_n \longrightarrow \mathbb{C}$ satisfying the following conditions (AF1)–(AF3):

(AF1) f is an eigenfunction for all $SL(n,\mathbb{R})$-invariant differential operators on $\mathfrak{P}_n$.
(AF2) $f(\gamma Y\, {}^t\gamma) = f(Y)$ for all $\gamma \in \Gamma_n$ and $Y \in \mathfrak{P}_n$.
(AF3) There exist a constant $C > 0$ and $s \in \mathbb{C}^{n-1}$ with $s = (s_1, \cdots, s_{n-1})$

such that $|f(Y)| \le C\,|p_{-s}(Y)|$ as the upper left determinants $\det Y_j \longrightarrow \infty$, $j = 1, 2, \cdots, n-1$, where

$$p_{-s}(Y) := \prod_{j=1}^{n-1} (\det Y_j)^{-s_j}$$

is Selberg's power function (cf. [3,6]).

We denote by $\mathbf{A}(\Gamma_n)$ the space of all automorphic forms for Γ_n. A cusp form $f \in \mathbf{A}(\Gamma_n)$ is an automorphic form for Γ_n satisfying the following conditions:

$$\int_{X\in(\mathbb{R}/\mathbb{Z})^{(j,n-j)}} f\left(Y\left[\begin{pmatrix} I_j & X \\ 0 & I_{n-j}\end{pmatrix}\right]\right) dX = 0, \quad 1 \le j \le n-1.$$

Here, $(\mathbb{R}/\mathbb{Z})^{(j,n-j)}$ denotes the set of all $j \times (n-j)$ matrices with entries in the one-dimensional real torus $\mathbb{R}/\mathbb{Z}$. We denote by $\mathbf{A}_0(\Gamma_n)$ the space of all cusp forms for Γ_n.

Definition 4. *For any positive integer $n \ge 2$, we define $\mathfrak{H}_n$ to be the set of all $n \times n$ real matrices of the form $z = x \cdot y$, where*

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{22} & \cdots & x_{2n} \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$y = \mathrm{diag}(y_1 y_2 \cdots y_{n-1}, y_1 y_2 \cdots y_{n-2}, \cdots, y_1, 1)$$

with $x_{ij} \in \mathbb{R}$ for $1 \le i < j \le n$ and $y_i \ge 0$ for $1 \le i \le n-1$.

We can show that $\mathfrak{H}_n$ is diffeomorphic to $\mathfrak{P}_n$. In fact, we have the Iwasawa decomposition

$$GL(n,\mathbb{R}) = \mathfrak{H}_n \cdot O(n) \cdot Z_n,$$

where $Z_n (\cong \mathbb{R}^\times)$ is the center of $GL(n,\mathbb{R})$ (cf. [7], (Proposition 1.2.6, pp. 11–12)). Here,

$$O(n) := O(n,\mathbb{R}) = \{k \in GL(n,\mathbb{R}) \mid {}^t k k = k\, {}^t k = I_n\}$$

denotes the real orthogonal group of degree n. We see easily that

$$\mathfrak{H}_n \cong GL(n,\mathbb{R})/(O(n) \cdot \mathbb{R}^\times),$$

where $\cong$ denotes the diffeomorphism.

It is seen that $GL(n,\mathbb{R})$ acts on $\mathfrak{H}_n$ by left translation (cf. [7], (Proposition 1.2.10, p. 14)). Then, we obtain

$$SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n) \cong SL(n,\mathbb{Z})\backslash GL(n,\mathbb{R})/(O(n)\cdot \mathbb{R}^\times),$$

where $SO(n) := SO(n,\mathbb{R}) = SL(n,\mathbb{R}) \cap O(n)$. We let

$$\mathfrak{S}_n := SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n) = SL(n,\mathbb{Z})\backslash \mathfrak{P}_n, \quad \mathfrak{P}_n := SL(n,\mathbb{R})/SO(n)$$

be the locally symmetric space. Therefore, we obtain the following isomorphism:

$$\mathfrak{S}_n \cong SL(n,\mathbb{Z})\backslash \mathfrak{H}_n.$$

$\mathfrak{P}_n$ parameterizes special principally polarized real tori of dimension n (cf. [15]). The arithmetic quotient $\mathfrak{S}_n$ is the moduli space of isomorphism classes of special principally polarized real tori of dimension n. Unfortunately, $\mathfrak{S}_n$ does not admit the structure of a real algebraic variety and does not admit a compactification which is defined over the rational number field $\mathbb{Q}$ (cf. [16] or [17]).

Remark 7. *In [25,26], Borel and Ji constructed the geodesic compactification, the standard compactification, and a maximal Satake compactification of the locally symmetric space $\mathfrak{S}_n$.*

Remark 8. *Müller [27] studied Weyl's law for the cuspidal spectrum of $SL(n, \mathbb{R})$. In [28], Lapid and Müller studied the cuspidal spectrum of $\mathfrak{S}_n$. In [29], Matz and Müller introduced the analytic torsion for $\mathfrak{S}_n$.*

Proposition 1. *We let $n \geq 2$. Following the coordinates of Definition 4, we put*

$$d^*x = \prod_{1 \leq i < j \leq n} dx_{ij} \quad \text{and} \quad dy^* = \prod_{k=1}^{n-1} y_k^{-n(n-k)-1} dy_k.$$

Then,

$$d^*z = d^*x \cdot d^*y$$

is the left $SL(n, \mathbb{R})$-invariant volume element on $\mathfrak{H}_n$.

Proof. The proof can be found in [7] (Proposition 1.5.3, pp. 25–26). □

Theorem 10. *We let $n \geq 2$. Then, volume $\mathbf{Vol}(\Gamma^n \backslash \mathfrak{H}_n)$ of $\Gamma^n \backslash \mathfrak{H}_n$ is given by*

$$\mathbf{Vol}(\Gamma^n \backslash \mathfrak{H}_n) = \int_{\Gamma^n \backslash \mathfrak{H}_n} d^*z = n \cdot 2^{n-1} \cdot \prod_{k=2}^{n} \frac{\zeta(k)}{\mathbf{Vol}(S^{k-1})},$$

where $\Gamma^n = SL(n, \mathbb{Z})$ and

$$\mathbf{Vol}(S^{k-1}) = \frac{2(\sqrt{\pi})^k}{\Gamma(k/2)}$$

denotes the volume of the $(k-1)$-dimensional sphere S^{k-1}, $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ is the Riemann zeta function, and $\Gamma(p)$ denotes the usual Gamma function.

Proof. The proof can be found in [7] (Theorem 1.6.1, pp. 27–37). □

Remark 9. *Since $\mathfrak{H}_n$ is diffeomorphic to $\mathfrak{P}_n$,*

$$\mathbf{Vol}(\Gamma^n \backslash \mathfrak{H}_n) = \mathbf{Vol}(\Gamma^n \backslash \mathfrak{P}_n) \qquad \text{(see (31))}.$$

The calculation of Goldfeld [7] (Theorem 1.6.1, pp. 27–37) is different from that of Garret [24].

For any $\nu = (\nu_1, \nu_2, \cdots, \nu_{n-1})$, we define the function $I_\nu : \mathfrak{H}_n \longrightarrow \mathbb{C}$ by

$$I_\nu(z) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{ij}\nu_j}, \tag{41}$$

where

$$b_{ij} := \begin{cases} ij, & \text{if } i + j \leq n \\ (n-i)(n-j), & \text{if } i + j \geq n. \end{cases}$$

Then, we see that $I_\nu(z)$ is an eigenfunction of $\mathbb{D}(\mathfrak{H}_n)$. Let us write

$$DI_\nu(z) = \lambda_D \cdot I_\nu(z) \quad \text{for every } D \in \mathbb{D}(\mathfrak{H}_n) \tag{42}$$

since

$$\lambda_{D_1 D_2} = \lambda_{D_1} \lambda_{D_2} \qquad \text{for all } D_1, D_2 \in \mathbb{D}(\mathfrak{H}_n).$$

Function λ_D (viewed as a function of D) is a character of $\mathbb{D}(\mathfrak{H}_n)$ which is called the Harish–Chandra character.

Following Goldfeld (cf. [7], (Definition 5.1.3, pp. 115–116), the notion of a Maass form is defined in the following way.

Definition 5. *We let $n \geq 2$. We put $\Gamma^n = SL(n, \mathbb{Z})$. For any $\nu = (\nu_1, \nu_2, \cdots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, a smooth $f : \Gamma^n \backslash \mathfrak{H}_n \longrightarrow \mathbb{C}$ is said to be a* Maass form *for Γ^n of type ν if it satisfies the following conditions* (M1)–(M3)*:*

(M1) $F(\gamma z) = f(z)$ for all $\gamma \in \Gamma^n$and $z \in \mathfrak{H}_n$.
(M2) $Df(z) = \lambda_D f(z)$ for all $D \in \mathbb{D}(\mathfrak{H}_n)$ given by (42).
(M3) $\int_{\Gamma^n \cap U \backslash U} f(uz) du = 0$ *for all upper triangular groups U of the form*

$$U = \left\{ \begin{pmatrix} I_{r_1} & * & * & * \\ 0 & I_{r_2} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & I_{r_b} \end{pmatrix} \right\}$$

with $r_1 + r_2 + \cdots + r_b = n$. Here, I_r denotes the $r \times r$ identity matrix and $*$ denotes arbitrary real matrices.

Remark 10. *In [7], Dorian Goldfeld studied Whittaker functions associated with Maass forms, Hecke operators for Γ^n, the Godement-Jacquet L-function for Γ^n, Eisenstein series for Γ^n, and Poincaré series for Γ^n.*

5. Invariant Differential Operators on $Sp(2n, \mathbb{R})/U(n)$

The first part of this section is based on the author's paper [30] (pp. 279–281). Throughout this section, we let $G := Sp(2n, \mathbb{R})$ and $K = U(n)$. We let

$$\mathbb{H}_n := \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im} \Omega > 0 \,\}$$

be the Siegel upper half plane of degree n. Then, G acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{43}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $\Omega \in \mathbb{H}_n$. The stabilizer of Action (43) at iI_n is

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \Big|\ A + iB \in U(n) \right\} \cong U(n).$$

Thus, we obtain the biholomorphic map

$$G/K \longrightarrow \mathbb{H}_n, \qquad gK \mapsto g \cdot iI_n, \quad g \in G.$$

$\mathbb{H}_n$ is a Hermitian symmetric manifold. In fact, it is known that $\mathbb{H}_n$ is an Einstein–Kähler Hermitian symmetric space.

For $\Omega = (\omega_{ij}) \in \mathbb{H}_n$, we write $\Omega = X + i\,Y$ with $X = (x_{ij}),\ Y = (y_{ij})$ real. We put $d\Omega = (d\omega_{ij})$ and $d\overline{\Omega} = (d\overline{\omega}_{ij})$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{\Omega}_{ij}} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \overline{\omega}_{ij}} \right).$$

C. L. Siegel [31] introduced the symplectic metric $ds^2_{n;A}$ on $\mathbb{H}_n$ invariant under the action (43) of $Sp(2n,\mathbb{R})$ that is given by

$$ds^2_{n;A} = A \cdot \mathrm{Tr}(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}), \qquad A > 0. \tag{44}$$

It is known that the metric $ds^2_{n;A}$ is a Kähler–Einstein metric. H. Maass [32] proved that its Laplace operator $\Delta_{n;A}$ is given by

$$\Delta_{n;A} = \frac{4}{A} \cdot \mathrm{Tr}\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right) \tag{45}$$

and

$$dv_n(\Omega) = (\det Y)^{-(n+1)} \prod_{1 \le i \le j \le n} dx_{ij} \prod_{1 \le i \le j \le n} dy_{ij} \tag{46}$$

is a $Sp(2n,\mathbb{R})$-invariant volume element on $\mathbb{H}_n$ (cf. [33], (p. 130)).

We let $\mathbb{D}(\mathbb{H}_n)$ be the algebra of all differential operators on $\mathbb{H}_n$ invariant under Action (43). Then, according to Harish–Chandra [10,11],

$$\mathbb{D}(\mathbb{H}_n) = \mathbb{C}[D_1, \cdots, D_n],$$

where $D_1, \cdots, D_n$ are algebraically independent invariant differential operators on $\mathbb{H}_n$. That is, $\mathbb{D}(\mathbb{H}_n)$ is a commutative algebra that is finitely generated by n algebraically independent invariant differential operators on $\mathbb{H}_n$. Maass [2] found the explicit $D_1, \cdots, D_n$. We let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of the Lie algebra of G. It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ (cf. [4]).

Now, we review differential operators on the Siegel upper half plane $\mathbb{H}_n$ invariant under Action (43). The isotropy subgroup K at iI_n for Action (43) is a maximal compact subgroup given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\Big|\, A\,{}^tA + B\,{}^tB = I_n,\ A\,{}^tB = B\,{}^tA,\ A, B \in \mathbb{R}^{(n,n)} \right\} \cong U(n).$$

We let $\mathfrak{k}$ be the Lie algebra of K. Then, the Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{pmatrix} \,\Big|\, X_1, X_2, X_3 \in \mathbb{R}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\Big|\, {}^tX + X = 0,\ Y = {}^tY \right\},$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \,\Big|\, X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

The subspace $\mathfrak{m}$ of $\mathfrak{g}$ may be regarded as the tangent space of $\mathbb{H}_n$ at iI_n. The adjoint representation of G on $\mathfrak{g}$ induces the action of K on $\mathfrak{m}$ given by

$$k \circ Z = kZ\,{}^tk, \quad \text{where } k \in K \text{ and } Z \in \mathfrak{m}. \tag{47}$$

We let T_n be the vector space of $n \times n$ symmetric complex matrices. We let $\Psi : \mathfrak{m} \longrightarrow T_n$ be the map defined by

$$\Psi\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}\right) = X + i\,Y, \quad \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{m}. \tag{48}$$

We let $\delta : K \longrightarrow U(n)$ be the isomorphism defined by

$$\delta\left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\right) = A + i\,B, \quad \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K, \tag{49}$$

where $U(n)$ denotes the unitary group of degree n. We identify $\mathfrak{m}$ (resp. K) with T_n (resp. $U(n)$) through the map Ψ (resp. δ). We consider the action of $U(n)$ on T_n defined by

$$h \cdot \omega = h\omega\, {}^t h, \quad h \in U(n),\ \omega \in T_n. \tag{50}$$

Then, the adjoint Action (47) of K on $\mathfrak{m}$ is compatible with Action (50) of $U(n)$ on T_n through map Ψ. Precisely for any $k \in K$ and $Z \in \mathfrak{m}$, we obtain

$$\Psi(k\,Z\,{}^t k) = \delta(k)\,\Psi(Z)\,{}^t\delta(k). \tag{51}$$

Action (50) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}(T_n)$ and the symmetric algebra $S(T_n)$, respectively. We denote by $\mathrm{Pol}(T_n)^{U(n)}$ $\left(\text{resp. } S(T_n)^{U(n)}\right)$ the subalgebra of $\mathrm{Pol}(T_n)$ $\left(\text{resp. } S(T_n)\right)$ consisting of $U(n)$-invariants. The following inner product $(\,,\,)$ on T_n defined by

$$(Z, W) = \mathrm{Tr}\left(Z\,\overline{W}\right), \quad Z, W \in T_n$$

gives an isomorphism as vector spaces

$$T_n \cong T_n^*, \quad Z \mapsto h_Z, \quad Z \in T_n, \tag{52}$$

where T_n^* denotes the dual space of T_n and f_Z is the linear functional on T_n defined by

$$h_Z(W) = (W, Z), \quad W \in T_n.$$

It is known that there is a canonical linear bijection of $S(T_n)^{U(n)}$ onto the algebra $\mathbb{D}(\mathbb{H}_n)$ of differential operators on $\mathbb{H}_n$ invariant under Action (43) of G. Identifying T_n with T_n^* by the above isomorphism (52), we obtain a canonical linear bijection

$$\Theta_n : \mathrm{Pol}(T_n)^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_n) \tag{53}$$

of $\mathrm{Pol}(T_n)^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_n)$. The map Θ_n is described explicitly as follows. Similarly, Action (47) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{m})$ and the symmetric algebra $S(\mathfrak{m})$, respectively. Through map Ψ, the subalgebra $\mathrm{Pol}(\mathfrak{m})^K$ of $\mathrm{Pol}(\mathfrak{m})$ consisting of K-invariants is isomorphic to $\mathrm{Pol}(T_n)^{U(n)}$. We put $N_* = n(n+1)$. We let $\{\xi_\alpha \mid 1 \le \alpha \le N_*\}$ be a basis of a real vector space $\mathfrak{m}$. If $P \in \mathrm{Pol}(\mathfrak{m})^K$, then

$$\left(\Theta_n(P)f\right)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \cdot \exp\left(\sum_{\alpha=1}^{N_*} t_\alpha \xi_\alpha\right) K\right)\right]_{(t_\alpha)=0}, \tag{54}$$

where $f \in C^\infty(\mathbb{H}_n)$. We refer to [4] (Theorem 4.6, pp. 285–286) for more detail. In general, it is hard to express $\Theta_n(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{m})^K$.

According to the work of Harish–Chandra [10,11], the algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators and is isomorphic to the commutative algebra $\mathbb{C}[x_1, \cdots, x_n]$ with n indeterminates. We note that n is the real rank of G. We let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. H. Maass found algebraically independent generators $D_1, D_2, \cdots, D_n$ of $\mathbb{D}(\mathbb{H}_n)$ ([2], (pp. 112–118)). In fact, we see that

$$-D_1 = \Delta_{n;1} = 4\,\mathrm{Tr}\left(Y\,{}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right) \tag{55}$$

is the Laplace operator for the invariant metric $ds^2_{n;1}$ on $\mathbb{H}_n$. Shimura [8] found another algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$.

Example 1. *We consider the case when $n = 1$. The algebra $\mathrm{Pol}(T_1)^{U(1)}$ is generated by the polynomial*

$$q(\omega) = \omega\,\overline{\omega}, \quad \omega = x + iy \in \mathbb{C} \text{ with } x, y \text{ real}.$$

Using Formula (54), we obtain

$$\Theta_1(q) = 4\,y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Therefore, $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}\big[\Theta_1(q)\big] = \mathbb{C}[D_1]$.

For two complex numbers $\alpha, \beta \in \mathbb{C}$, Maass considered the following matrix-valued differential operator given by

$$\Omega_{\alpha,\beta} := \Lambda_{\beta - \frac{n+1}{2}} K_\alpha + \alpha \left(\beta - \frac{n+1}{2} \right) \cdot I_n, \tag{56}$$

where

$$K_\alpha := (\Omega - \overline{\Omega}) \frac{\partial}{\partial \Omega} + \alpha \cdot I_n$$

and

$$\Lambda_{\beta - \frac{n+1}{2}} := (\Omega - \overline{\Omega}) \frac{\partial}{\partial \overline{\Omega}} - \left(\beta - \frac{n+1}{2} \right) \cdot I_n.$$

That is,

$$\begin{aligned} \Omega_{\alpha,\beta} &= (\Omega - \overline{\Omega})\,{}^t\!\left((\Omega - \overline{\Omega}) \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} - \beta (\Omega - \overline{\Omega}) \frac{\partial}{\partial \Omega} + \alpha (\Omega - \overline{\Omega}) \frac{\partial}{\partial \overline{\Omega}} \\ &= -4 \cdot Y\,{}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} - 2\,\beta\, i\, Y \frac{\partial}{\partial \Omega} + 2\,\alpha\, i\, Y \frac{\partial}{\partial \overline{\Omega}}. \end{aligned}$$

We refer to [32] (p. 49), [34] (p. 176), and [2], (p. 119). Then,

$$\mathrm{Tr}(\Omega_{\alpha,\beta}) = -\Delta_{n;1} - 2\,\beta\, i\, \mathrm{Tr}\left(Y \frac{\partial}{\partial \Omega} \right) + 2\,\alpha\, i\, \mathrm{Tr}\left(Y \frac{\partial}{\partial \overline{\Omega}} \right), \tag{57}$$

where $\Delta_{n;1}$ is the Laplace operator of $(\mathbb{H}_n, ds^2_{n;1})$ (see Formulas (44), (45) and (55)).

Definition 6. *The differential operator*

$$\mathscr{L}_{\alpha,\beta} := -\mathrm{Tr}(\Omega_{\alpha,\beta})$$

is called the Siegel–Maass Laplacian *of weight (α, β).*

Remark 11. *We note that* $\mathscr{L}_{0,0} = \Delta_{n;1} \in \mathbb{D}(\mathbb{H}_n)$ *but* $\mathscr{L}_{\alpha,\beta} \notin \mathbb{D}(\mathbb{H}_n)$ *if* $(\alpha, \beta) \neq (0, 0)$.

The following definition is given by Kramer and Mandal (cf. [9], (Definition 4.7]).

Definition 7. *We set* $\Gamma_n^\flat = Sp(2n, \mathbb{Z})$. *We let* $\Gamma \subset Sp(2n, \mathbb{R})$ *be a subgroup of* $Sp(2n, \mathbb{R})$ *commensurable with* $\Gamma_n^\flat$, *i.e., the intersection* $\Gamma \cap \Gamma_n^\flat$ *is a finite index subgroup of* Γ *as well as of* $\Gamma_n^\flat$. *We let* $\gamma_j \in \Gamma_n^\flat$ $(1 \le j \le h)$ *denote a set of representatives for the left cosets of* $\Gamma \cap \Gamma_n^\flat$ *in* $\Gamma_n^\flat$. *For two complex numbers* $\alpha, \beta \in \mathbb{C}$, *we then let* $\mathfrak{V}^n_{\alpha,\beta}(\Gamma)$ *denote the space of all functions* $\varphi : \mathbb{H}_n \longrightarrow \mathbb{C}$ *satisfying the following conditions* (KM1)–(KM3)*:*

(KM1) *φ is real analytic;*

(KM2) $\varphi(\gamma \cdot \Omega) = \det(C\Omega + D)^\alpha \det(C\overline{\Omega} + D)^\beta \varphi(\Omega)$ for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$;

(KM3) *given* $Y_0 \in \mathbb{R}^{(n,n)}$ *with* $Y_0 = {}^tY_0 > 0$, *there exist a positive real number* $M \in \mathbb{R}^+$ *and a positive integer* $N \in \mathbb{Z}^+$ *such that the inequalities*

$$|\det(C_j\Omega + D_j)^{-\alpha} \det(C_j\overline{\Omega} + D_j)^{-\beta} \varphi(\gamma_j \cdot \Omega)| \le M \cdot \mathrm{Tr}(Y)^N$$

holds in the region $\{\Omega = X + i\,Y \in \mathbb{H}_n \mid Y \ge Y_0\}$ *for the set of representatives*

$$\gamma_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \Gamma_n^\flat \, (1 \le j \le h).$$

Remark 12. *For* $\varphi \in \mathfrak{V}^n_{\alpha,\beta}(\Gamma)$, *we set*

$$\| \varphi \|_{\alpha,\beta} = \int_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta} |\varphi(\Omega)|^2 \, dv_n(\Omega),$$

where $dv_n(\Omega)$ *is a* $Sp(2n, \mathbb{R})$*-invariant volume element on* $\mathbb{H}_n$ (see Formula (46)). *In this way, we obtain the Hilbert space*

$$\mathscr{H}^n_{\alpha,\beta}(\Gamma) := \{ \varphi \in \mathfrak{V}^n_{\alpha,\beta}(\Gamma) \mid \| \varphi \|_{\alpha,\beta} < \infty \}$$

equipped with the inner product

$$\langle \varphi, \psi \rangle_{\alpha,\beta} := \int_{\Gamma\backslash\mathbb{H}_n} \det(Y)^{\alpha+\beta} \varphi(\Omega)\, \overline{\psi(\Omega)}\, dv_n(\Omega)$$

for all $\varphi, \psi \in \mathscr{H}^n_{\alpha,\beta}(\Gamma)$. *We note that in order to enable* $\|\varphi\| < \infty$, *the exponent* $N \in \mathbb{Z}^+$ *in part* (KM3) *of* Definition 7 *has to be* 0.

Remark 13. *Kramer and Mandal showed that the Siegel–Maass Laplacian* $\mathscr{L}_{\alpha,\beta}$ *acts as a symmetric operator on* $\mathscr{H}^n_{\alpha,\beta}(\Gamma)$ *(cf. [9], (Theorem 5.1, pp. 11–15). That is,*

$$\langle \mathscr{L}_{\alpha,\beta}\varphi, \psi \rangle_{\alpha,\beta} = \langle \varphi, \mathscr{L}_{\alpha,\beta}\psi \rangle_{\alpha,\beta}$$

for all $\varphi, \psi \in \mathscr{H}^n_{\alpha,\beta}(\Gamma)$.

Definition 8. *We let* $\Gamma \subset Sp(2n, \mathbb{R})$ *be a subgroup of* $Sp(2n, \mathbb{R})$ *commensurable with* $\Gamma_n^\flat$. *The elements of* $\mathscr{H}^n_{\alpha,\beta}(\Gamma)$ *are called* automorphic forms of weight *(α, β) and degree n for* Γ. *Moreover, if* $\varphi \in \mathscr{H}^n_{\alpha,\beta}(\Gamma)$ *is an eigenfunction of* $\mathscr{L}_{\alpha,\beta}$, *it is called a* Siegel–Maass form of weight *(α, β) and degree n for* Γ.

For the present being, we assume that $\Gamma \subset Sp(2n,\mathbb{R})$ is a subgroup of $Sp(2n,\mathbb{R})$ commensurable with $\Gamma_n^\flat$. The case that $\alpha = \frac{k}{2}$ and $\beta = -\frac{k}{2}$ with $k \in \mathbb{Z}^+$ provides an application to the study of Siegel cusp forms of weight k for Γ. We recall the notion of Siegel modular forms.

Definition 9. *We let $\Gamma \subset Sp(2n,\mathbb{R})$ be a subgroup of $Sp(2n,\mathbb{R})$ commensurable with $\Gamma_n^\flat$. We let $\gamma_j \in \Gamma_n^\flat$ $(1 \le j \le h)$ denote a set of representatives for the left cosets of of $\Gamma \cap \Gamma_n^\flat$ in $\Gamma_n^\flat$. Function $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ is called a* Siegel modular form of weight *k and degree n for Γ if it satisfies the following conditions* (SI1)–(SI3)*:*

(SI1) *f is holomorphic;*

(SI2) $f(\gamma \cdot \Omega) = \det(C\Omega + D)^k f(\Omega) \quad$ for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$*;*

(SI3) *given $Y_0 \in \mathbb{R}^{(n,n)}$ with $Y_0 = {}^tY_0 > 0$, the quantities* $\det(C_j\Omega + D_j)^{-k} f(\gamma_j \cdot \Omega)$ *are bounded in the region $\{\Omega = X + i\,Y \in \mathbb{H}_n \mid Y \ge Y_0\}$ for the set of representatives $\gamma_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \Gamma_n^\flat$ $(1 \le j \le h)$.*

We denote by $M_n(\Gamma, k)$ the vector space of all Siegel modular form of weight k and degree n for Γ. It is known that $M_n(\Gamma,k)$ is finite dimensional. Moreover, a Siegel modular form $f \in M_n(\Gamma,k)$ is called a Siegel cusp form of weight *k and degree n for Γ if the condition* (SI3) *is strengthened to the following condition* (SI3)**:*

(SI3)* *given $Y_0 \in \mathbb{R}^{(n,n)}$ with $Y_0 = {}^tY_0 > 0$, the quantities* $\det(C_j\Omega + D_j)^{-k} f(\gamma_j \cdot \Omega)$ *become arbitrarily small in the region $\{\Omega = X + i\,Y \in \mathbb{H}_n \mid Y \ge Y_0\}$ for the set of representatives $\gamma_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in \Gamma_n^\flat$ $(1 \le j \le h)$.*

We denote by $\mathfrak{C}_k^n(\Gamma)$ the vector space of Siegel cusp form of weight k and degree n for Γ. The vector space $\mathfrak{C}_k^n(\Gamma)$ is a Hermitian inner product space equipped with the Petersson inner product given by

$$\langle f, g \rangle := \int_{\Gamma\backslash\mathbb{H}_n} \det(Y)^k\, f(\Omega)\, \overline{g(\Omega)}\, dv_n(\Omega) \qquad (\, f, g \in \mathfrak{C}_k^n(\Gamma)\,).$$

We have

$$\Omega_{\frac{k}{2},-\frac{k}{2}} = -4 \cdot Y \, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}}\right) \frac{\partial}{\partial \Omega} + i\,k\,Y \frac{\partial}{\partial X} \qquad \text{and} \qquad \mathscr{L}_{\frac{k}{2},-\frac{k}{2}} = \mathrm{Tr}(\Omega_{\frac{k}{2},-\frac{k}{2}}). \tag{58}$$

Theorem 11. *We let $\Gamma \subset Sp(2n,\mathbb{R})$ be a subgroup of $Sp(2n,\mathbb{R})$ commensurable with $\Gamma_n^\flat$ and let $\varphi \in \mathscr{H}^n_{\frac{k}{2},-\frac{k}{2}}(\Gamma)$ be a Siegel–Maass form of weight $(\frac{k}{2}, -\frac{k}{2})$ for Γ. Then, if $\mathscr{L}_{\frac{k}{2},-\frac{k}{2}}\varphi = \lambda\,\varphi$, then $\lambda \in \mathbb{R}$ and*

$$\lambda \ge \frac{nk}{4}\,(n+1-k).$$

The equality holds if and only if $\varphi(\Omega) = \det(Y)^{k/2} f(\Omega)$ for some Siegel cusp form $f \in \mathfrak{C}_k^n(\Gamma)$ of weight k for Γ. In other words,

$$\mathfrak{C}_k^n(\Gamma) \cong \mathrm{Ker}\left(\mathscr{L}_{\frac{k}{2},-\frac{k}{2}} + \frac{nk}{4}\,(n+1-k) \cdot I_n\right) \tag{59}$$

of complex vector spaces induced by the assignment

$$f(\Omega) \longmapsto \det(Y)^{k/2} f(\Omega), \qquad (\, \Omega = X + i\,Y \in \mathbb{H}_n,\ f \in \mathfrak{C}_k^n(\Gamma)\,).$$

Proof. The proof can be found in [9] (pp. 15–19). $\square$

Using the commutative algebra $\mathbb{D}(\mathbb{H}_n)$, we introduce the notion of Maass–Siegel function for Γ_n.

Definition 10. *Function* $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ *is said to be a* Maass–Siegel function *for* $\Gamma_n^\flat$ *if it satisfies the conditions* (MS1)–(MS4)*:*

(MS1) f *is real analytic;*
(MS2) $f(\gamma \cdot \Omega) = f(\Omega)$ *for all* $\gamma \in \Gamma_n^\flat$ *and* $\Omega \in \mathbb{H}_n$*;*
(MS3) f *is an eigenfunction of all invariant differential operators in* $\mathbb{D}(\mathbb{H}_n)$*;*
(MS4) *given* $Y_0 \in \mathbb{R}^{(n,n)}$ *with* $Y_0 = {}^tY_0 > 0$*, the quantities* $f(\Omega)$ *are bounded in the region* $\{\Omega = X + i\,Y \in \mathbb{H}_n \mid Y \geq Y_0\}$.

Function $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ *is said to be a* weak Maass–Siegel function *for* $\Gamma_n^\flat$ *if it satisfies the above conditions* (MS1), (MS2) *and* (MS4) *together with* (MS3)**:*

(MS3)* f *is an eigenfunction of the Laplace operator* $\Delta_{n;A}$ *of* $(\mathbb{H}_n, ds^2_{n;A})$.

Problem 1. *Develop the theory of harmonic analysis of* $L^2(\Gamma_n^\flat\backslash\mathbb{H}_n)$*. Develop the spectral theory of the Laplace operator* $\Delta_{n;A}$ *(see Formula (45)) on* $L^2(\Gamma_n^\flat\backslash\mathbb{H}_n)$*.*

6. Invariant Differential Operators on $GL_{n,m}/O(n,\mathbb{R})$

This section is based on papers [15,35]. We recall that the group

$$GL_{n,m}(\mathbb{R}) := GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

is the semidirect product of $GL(n,\mathbb{R})$ and the additive group $\mathbb{R}^{(m,n)}$ endowed with multiplication law

$$(g,\alpha) \circ (h,\beta) := (gh, \alpha\,{}^th^{-1} + \beta) \tag{60}$$

for all $g,h \in GL(n,\mathbb{R})$ and $\alpha,\beta \in \mathbb{R}^{(m,n)}$. We also recall the Minkowski–Euclid space

$$\mathscr{P}_{n,m} := \mathscr{P}_n \times \mathbb{R}^{(m,n)} = \{\, Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \,\} \times \mathbb{R}^{(m,n)}.$$

Then, $GL_{n,m}(\mathbb{R})$ acts on $\mathscr{P}_{n,m}$ naturally and transitively by

$$(g,\alpha)\cdot(Y,V) := (gY\,{}^tg, (V+\alpha)\,{}^tg) \tag{61}$$

for all $(g,\alpha) \in GL_{n,m}(\mathbb{R})$ and $(Y,V) \in \mathscr{P}_{n,m}$. Since $O(n,\mathbb{R})$ is the stabilizer of the action (61) at $(I_n, 0)$, the non-symmetric homogeneous space $GL_{n,m}(\mathbb{R})/O(n,\mathbb{R})$ is diffeomorphic to the Minkowski–Euclid space $\mathscr{P}_{n,m}$. We denote by $\mathbb{D}(\mathscr{P}_{n,m})$ the algebra of all differential operators on $\mathscr{P}_{n,m}$ invariant under Action (6.2) of $GL_{n,m}(\mathbb{R})$. We let

$$GL_{n,m}(\mathbb{Z}) := GL(n,\mathbb{Z}) \ltimes \mathbb{Z}^{(m,n)}$$

denote the discrete subgroup of $GL_{n,m}(\mathbb{R})$.

For a variable $(Y,V) \in \mathscr{P}_{n,m}$ with $Y \in \mathscr{P}_n$ and $V \in \mathbb{R}^{(m,n)}$, we put

$$Y = (y_{ij}) \text{ with } y_{ij} = y_{ji},\;\; V = (v_{kl}),$$

$$dY = (dy_{ij}),\;\; dV = (dv_{kl}),$$

$$[dY] = \bigwedge_{i\leq j} dy_{ij}, \qquad [dV] = \bigwedge_{k,l} dv_{kl},$$

and
$$\frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right), \quad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}} \right),$$
where $1 \leq i, j, l \leq n$ and $1 \leq k \leq m$.

For a fixed element $(g, \alpha) \in GL_{n,m}(\mathbb{R})$, we write
$$(Y_\star, V_\star) = (g, \alpha) \cdot (Y, V) = \big(g\, Y\, {}^t g, (V + \alpha)\, {}^t g\big),$$
where $(Y, V) \in \mathscr{P}_{n,m}$. Then, we obtain
$$Y_\star = g\, Y\, {}^t g, \quad V_\star = (V + \alpha)\, {}^t g \tag{62}$$
and
$$\frac{\partial}{\partial Y_\star} = {}^t g^{-1} \frac{\partial}{\partial Y} g^{-1}, \quad \frac{\partial}{\partial V_\star} = \frac{\partial}{\partial V} g^{-1}. \tag{63}$$

Lemma 1. *For all two positive real numbers a and b, the following metric $ds^2_{n,m;a,b}$ on $\mathscr{P}_{n,m}$ defined by*
$$ds^2_{n,m;a,b} = a \cdot \mathrm{Tr}(Y^{-1} dY\, Y^{-1} dY) + b \cdot \mathrm{Tr}(Y^{-1}\, {}^t(dV)\, dV) \tag{64}$$
is a Riemannian metric on $\mathscr{P}_{n,m}$ which is invariant under Action (61) of $GL_{n,m}(\mathbb{R})$. The Laplacian $\Delta_{n,m;a,b}$ of $(\mathscr{P}_{n,m},\, ds^2_{n,m;a,b})$ is given by
$$\triangle_{n,m;a,b} = \frac{1}{a} \cdot \mathrm{Tr}\left(\left(Y \frac{\partial}{\partial Y} \right)^2 \right) - \frac{m}{2\,a} \mathrm{Tr}\left(Y \frac{\partial}{\partial Y} \right) + \frac{1}{b} \cdot \sum_{k \leq p} \left(\left(\frac{\partial}{\partial V} \right) Y \, {}^t\!\left(\frac{\partial}{\partial V} \right) \right)_{kp}.$$
Moreover, $\triangle_{n,m;a,b}$ is a differential operator of order 2 which is invariant under Action (61) of $GL_{n,m}(\mathbb{R})$.

Proof. The proof can be found in [15] (Lemma 8.1, p. 312). □

Lemma 2. *The following volume element $d\mu_{n,m}(Y, V)$ on $\mathscr{P}_{n,m}$ defined by*
$$d\mu_{n,m}(Y, V) = (\det\, Y)^{-\frac{n+m+1}{2}} [dY][dV] \tag{65}$$
is invariant under Action (61) of $GL_{n,m}(\mathbb{R})$.

Proof. The proof can be found in [15] (Lemma 8.2, pp. 312–313). □

The Lie algebra $\mathfrak{g}_\star$ of $GL_{n,m}(\mathbb{R})$ is given by
$$\mathfrak{g}_\star = \left\{ (X, Z) \mid X \in \mathbb{R}^{(n,n)},\ Z \in \mathbb{R}^{(m,n)} \right\}$$
equipped with the following Lie bracket:
$$\big[(X_1, Z_1),\, (X_2, Z_2)\big]_\star = \big([X_1, X_2]_0,\, Z_2\, {}^t X_1 - Z_1\, {}^t X_2\big),$$
where $[X_1, X_2]_0 = X_1 X_2 - X_2 X_1$ denotes the usual matrix bracket and $(X_1, Z_1), (X_2, Z_2) \in \mathfrak{g}_\star$. The adjoint representation $\mathrm{Ad}_\star$ of $GL_{n,m}(\mathbb{R})$ is given by
$$\mathrm{Ad}_\star\,((g, \lambda))(X, Z) = \big(gXg^{-1},\, (Z - \lambda\, {}^t X)\, {}^t g \big), \tag{66}$$

where $(g,\lambda) \in GL_{n,m}(\mathbb{R})$ and $(X,Z) \in \mathfrak{g}_\star$ and the adjoint representation $\mathrm{ad}_\star$ of $\mathfrak{g}_\star$ on $\mathfrak{g}_\star$ is given by

$$\mathrm{ad}_\star\big((X,Z)\big)\big((X_1,Z_1)\big) = \big[(X,Z),(X_1,Z_1)\big]_\star.$$

We see that the Killing form $B_\star$ of $\mathfrak{g}_\star$ is given by

$$B_\star\big((X_1,Z_1),(X_2,Z_2)\big) = (2\,n+m)\,\mathrm{Tr}(X_1X_2) - 2\,\mathrm{Tr}(X_1)\,\mathrm{Tr}(X_2).$$

We let

$$K_\star := \{\,(k,0) \in GL_{n,m}(\mathbb{R}) \mid k \in O(n,\mathbb{R})\,\} \cong O(n,\mathbb{R}).$$

Then, the Lie algebra $\mathfrak{k}_\star$ of $K_\star$ is

$$\mathfrak{k}_\star = \Big\{\,(X,0) \in \mathfrak{g}_\star \mid X + {}^tX = 0\,\Big\}.$$

We let $\mathfrak{p}_\star$ be the subspace of $\mathfrak{g}_\star$ defined by

$$\mathfrak{p}_\star = \Big\{\,(X,Z) \in \mathfrak{g}_\star \mid X = {}^tX \in \mathbb{R}^{(n,n)},\ Z \in \mathbb{R}^{(m,n)}\,\Big\}.$$

Then, we have the following relation:

$$[\mathfrak{k}_\star,\mathfrak{k}_\star] \subset \mathfrak{k}_\star \quad \text{and} \quad [\mathfrak{k}_\star,\mathfrak{p}_\star] \subset \mathfrak{p}_\star.$$

In addition, we have

$$\mathfrak{g}_\star = \mathfrak{k}_\star \oplus \mathfrak{p}_\star \quad (\text{the direct sum}).$$

$K_\star$ acts on $\mathfrak{p}_\star$ via the adjoint representation $\mathrm{Ad}_\star$ of $GL_{n,m}(\mathbb{R})$ by

$$k_\star \cdot (X,Z) = \big(kX\,{}^tk, Z\,{}^tk\big), \tag{67}$$

where $k_\star = (k,0) \in K_\star$ with $k \in O(n,\mathbb{R})$ and $(X,Z) \in \mathfrak{p}_\star$.

For brevity, we set $K = O(n,\mathbb{R})$. Then, Action (68) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p}_\star)$ of $\mathfrak{p}_\star$ and the symmetric algebra $S(\mathfrak{p}_\star)$. We denote by $\mathrm{Pol}(\mathfrak{p}_\star)^K$ (resp. $S(\mathfrak{p}_\star)^K$) the subalgebra of $\mathrm{Pol}(\mathfrak{p}_\star)$ (resp. $S(\mathfrak{p}_\star)$) consisting of all K-invariants. The following inner product $(\,,\,)_\star$ on $\mathfrak{p}_\star$ defined by

$$\big((X_1,Z_1),(X_2,Z_2)\big)_\star = \mathrm{Tr}(X_1X_2) + \mathrm{Tr}(Z_1\,{}^tZ_2), \quad (X_1,Z_1),(X_2,Y_2) \in \mathfrak{p}_\star$$

gives an isomorphism as vector spaces

$$\mathfrak{p}_\star \cong \mathfrak{p}_\star^*, \quad (X,Z) \mapsto f_{X,Z}, \quad (X,Z) \in \mathfrak{p}_\star, \tag{68}$$

where $\mathfrak{p}_\star^*$ denotes the dual space of $\mathfrak{p}_\star$ and $f_{X,Z}$ is the linear functional on $\mathfrak{p}_\star$ defined by

$$f_{X,Z}\big((X_1,Z_1)\big) = \big((X,Z),(X_1,Z_1)\big)_\star, \quad (X_1,Z_1) \in \mathfrak{p}_\star.$$

We let $\mathbb{D}(\mathscr{P}_{n,m})$ be the algebra of all differential operators on $\mathscr{P}_{n,m}$ that are invariant under Action (61) of $GL_{n,m}(\mathbb{R})$. It is known that there is a canonical linear bijection of $S(\mathfrak{p}_\star)^K$ onto $\mathbb{D}(\mathscr{P}_{n,m})$. Identifying $\mathfrak{p}_\star$ with $\mathfrak{p}_\star^*$ by the above Isomorphism (68), we obtain a canonical linear bijection

$$\Phi_{n,m} : \mathrm{Pol}(\mathfrak{p}_\star)^K \longrightarrow \mathbb{D}(\mathscr{P}_{n,m}) \tag{69}$$

of $\mathrm{Pol}(\mathfrak{p}_\star)^K$ onto $\mathbb{D}(\mathscr{P}_{n,m})$. The map $\Phi_{n,m}$ is described explicitly as follows. We put $N_\star = n(n+1)/2 + mn$. We let $\{\eta_\alpha \mid 1 \le \alpha \le N_\star\}$ be a basis of $\mathfrak{p}_\star$. If $P \in \mathrm{Pol}(\mathfrak{p}_\star)^K$, then

$$\Big(\Phi_{n,m}(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\cdot \exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha \eta_\alpha\right)K\right)\right]_{(t_\alpha)=0}, \tag{70}$$

where $f \in C^\infty(\mathscr{P}_{n,m})$. We refer to [4] (pp. 280–289). In general, it is very hard to express $\Phi_{n,m}(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p}_\star)^K$.

We take a coordinate (X, Z) in $\mathfrak{p}_\star$ such that

$$X = \begin{pmatrix} x_{11} & \frac{1}{2}x_{12} & \dots & \frac{1}{2}x_{1n} \\ \frac{1}{2}x_{12} & x_{22} & \dots & \frac{1}{2}x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}x_{1n} & \frac{1}{2}x_{2n} & \dots & x_{nn} \end{pmatrix} \in \mathfrak{p} \quad \text{and} \quad Z = (z_{kl}) \in \mathbb{R}^{(m,n)}.$$

Here,

$$\mathfrak{p} := \{\, X \in \mathbb{R}^{(n,n)} \mid X = {}^tX \,\}.$$

We define the polynomials α_j, $\beta_{pq}^{(k)}$, R_{jp}, and S_{jp} on $\mathfrak{p}_\star$ by

$$\begin{aligned} \alpha_j(X,Z) &= \mathrm{Tr}(X^j), \quad 1 \le j \le n, & (71)\\ \beta_{pq}^{(k)}(X,Z) &= (Z\,X^k\,{}^tZ)_{pq}, \quad 0 \le k \le n-1,\ 1 \le p \le q \le m, & (72)\\ R_{jp}(X,Z) &= \mathrm{Tr}(X^j({}^tZZ)^p), \quad 0 \le j \le n-1,\ 1 \le p \le m, & (73)\\ S_{jp}(X,Z) &= \det(X^j({}^tZZ)^p), \quad 0 \le j \le n-1,\ 1 \le p \le m, & (74) \end{aligned}$$

where $(ZX\,{}^tZ)_{pq}$ denotes the (p,q)-entry of $ZX\,{}^tZ$.

We propose the following natural problems.

Problem 2. *Find a complete list of explicit generators of* $\mathrm{Pol}(\mathfrak{p}_\star)^K$.

Problem 3. *Find all the relations among a set of generators of* $\mathrm{Pol}(\mathfrak{p}_\star)^K$.

Problem 4. *Find an easy or effective way to express the images of the above invariant polynomials under the Helgason map* $\Phi_{n,m}$ *explicitly.*

Problem 5. *Decompose* $\mathrm{Pol}(\mathfrak{p}_\star)^K$ *into* $O(n,\mathbb{R})$*-irreducibles.*

Problem 6. *Find a complete list of explicit generators of the algebra* $\mathbb{D}(\mathscr{P}_{n,m})$. *Or construct explicit* $GL_{n,m}(\mathbb{R})$*-invariant differential operators on* $\mathscr{P}_{n,m}$.

Problem 7. *Find all the relations among a set of generators of* $\mathbb{D}(\mathscr{P}_{n,m})$.

Problem 8. *Is* $\mathrm{Pol}(\mathfrak{p}_\star)^K$ *finitely generated? Is* $\mathbb{D}(\mathscr{P}_{n,m})$ *finitely generated?*

Problem 9. *Find the center* $\mathscr{Z}_{n,m}$ *of* $\mathbb{D}(\mathscr{P}_{n,m})$.

M. Itoh [36] proved the following theorem.

Theorem 12. $\mathrm{Pol}(\mathfrak{p}_\star)^K$ *is generated by* α_j $(1 \le j \le n)$ *and* $\beta_{pq}^{(k)}$ $(0 \le k \le n-1,\ 1 \le p \le q \le m)$.

Proof. We refer to [36] (Theorem 3.1). □

According to the above theorem, he solved Problem 2 and Problem 8. He also solved Problem 3 in [36] (Theorem 3.2).

We present some invariant differential operators on $\mathscr{P}_{n,m}$. We define the differential operators D_j, Ω_{pq}, and L_p on $\mathscr{P}_{n,m}$ by

$$D_j = \mathrm{Tr}\left(\left(2Y \frac{\partial}{\partial Y} \right)^j \right), \quad 1 \le j \le n, \tag{75}$$

$$\Omega_{pq}^{(k)} = \left\{ \frac{\partial}{\partial V} \left(2Y \frac{\partial}{\partial Y} \right)^k Y \, {}^t\!\left(\frac{\partial}{\partial V} \right) \right\}_{pq}, \quad 0 \le k \le n-1,\ 1 \le p \le q \le m \tag{76}$$

and

$$L_p = \mathrm{Tr}\left(\left\{ Y \, {}^t\!\left(\frac{\partial}{\partial V} \right) \frac{\partial}{\partial V} \right\}^p \right), \quad 1 \le p \le m. \tag{77}$$

Here, for matrix A we denote by A_{pq} the (p,q)-entry of A.

Also, we define the invariant differential operators $\mathfrak{S}_{jp}$ by

$$\mathfrak{S}_{jp} = \mathrm{Tr}\left(\left(2Y \frac{\partial}{\partial Y} \right)^j \left\{ Y \, {}^t\!\left(\frac{\partial}{\partial V} \right) \frac{\partial}{\partial V} \right\}^p \right), \tag{78}$$

where $1 \le j \le n$ and $1 \le p \le m$.

Remark 14. *It is seen that* $[D_1, \Omega_{pq}^{(0)}] = 2\,\Omega_{pq}^{(0)}$ (cf. *[35], (Theorem 8.1, p. 304)). Therefore,* $\mathbb{D}(\mathscr{P}_{n,m})$ *is not commutative. We refer to [35] for more details on invariant differential operators on the Minkowski–Euclid space* $\mathscr{P}_{n,m}$.

We want to mention the special invariant differential operator on $\mathscr{P}_{n,m}$. In [37], the author studied the following differential operator $M_{n,m;\mathscr{M}}$ on $\mathscr{P}_{n,m}$ defined by

$$M_{n,m;\mathscr{M}} = \det(Y) \cdot \det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi} \, {}^t\!\left(\frac{\partial}{\partial V} \right) \mathscr{M}^{-1} \left(\frac{\partial}{\partial V} \right) \right), \tag{79}$$

where $\mathscr{M}$ is a positive definite, symmetric half-integral matrix of degree m. This differential operator characterizes *singular Jacobi forms*. For more detail, we refer to [37]. According to (62) and (63), we see easily that the differential operator $M_{n,m;\mathscr{M}}$ is invariant under Action (61) of $GL_{n,m}(\mathbb{R})$.

Question: Calculate the inverse $\Phi_{n,m}^{-1}(M_{n,m;\mathscr{M}})$ of $M_{n,m;\mathscr{M}}$ under the Helgason map $\Phi_{n,m}$.

7. Invariant Differential Operators on $SL_{n,m}(\mathbb{R})/SO(n,\mathbb{R})$

We recall that the group

$$SL_{n,m}(\mathbb{R}) := SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

is the semidirect product of $SL(n,\mathbb{R})$ and the additive group $\mathbb{R}^{(m,n)}$ endowed with multiplication law

$$(g,\alpha) \circ (h,\beta) := (gh, \alpha \, {}^t h^{-1} + \beta) \tag{80}$$

for all $g, h \in SL(n,\mathbb{R})$, and $\alpha, \beta \in \mathbb{R}^{(m,n)}$. We also recall the homogeneous space

$$\mathfrak{P}_{n,m} := \mathfrak{P}_n \times \mathbb{R}^{(m,n)} = \{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0,\ \det Y = 1 \} \times \mathbb{R}^{(m,n)}.$$

Then, $SL_{n,m}(\mathbb{R})$ acts on $\mathfrak{P}_{n,m}$ naturally and transitively by

$$(g, \alpha) \cdot (Y, V) := (gY\,{}^t g, (V + \alpha)\,{}^t g) \tag{81}$$

for all $(g, \alpha) \in SL_{n,m}(\mathbb{R})$ and $(Y, V) \in \mathfrak{P}_{n,m}$. Since $SO(n, \mathbb{R})$ is the stabilizer of Action (81) at $(I_n, 0)$, the non-symmetric homogeneous space $SL_{n,m}(\mathbb{R})/SO(n, \mathbb{R})$ is diffeomorphic to the non-symmetric space $\mathfrak{P}_{n,m}$. We denote by $\mathbb{D}(\mathfrak{P}_{n,m})$ the algebra of all differential operators on $\mathfrak{P}_{n,m}$ invariant under Action (81) of $SL_{n,m}(\mathbb{R})$. We let

$$SL_{n,m}(\mathbb{Z}) := SL(n, \mathbb{Z}) \ltimes \mathbb{Z}^{(m,n)}$$

denote the discrete subgroup of $SL_{n,m}(\mathbb{R})$.

From now on, we write $G_\diamond = SL_{n,m}(\mathbb{R})$ for brevity. We let

$$\mathfrak{sl}(n, \mathbb{R}) = \left\{ X \in \mathbb{R}^{(n,n)} \mid \mathrm{Tr}(X) = 0 \right\}$$

be the Lie algebra of $SL(n, \mathbb{R})$. Then, it is easy to see that the Lie algebra $\mathfrak{g}_\diamond$ of $G_\diamond$ is given by

$$\mathfrak{g}_\diamond = \left\{ (X, Z) \mid X \in \mathfrak{sl}(n, \mathbb{R}),\ Z \in \mathbb{R}^{(m,n)} \right\} \tag{82}$$

equipped with the following Lie bracket:

$$[(X_1, Z_1), (X_2, Z_2)]_\diamond = ([X_1, X_2]_0,\ Z_2\,{}^t X_1 - Z_1\,{}^t X_2), \tag{83}$$

where $[X_1, X_2]_0 := X_1X_2 - X_2X_1$ denotes the usual matrix bracket and (X_1, Z_1), (X_2, Z_2) $\in \mathfrak{g}_\diamond$. The adjoint representation $\mathrm{Ad}_\diamond$ of $G_\diamond$ is given by

$$\mathrm{Ad}_\diamond((g, \alpha))(X, Z) = (gXg^{-1},\ (Z - \alpha\,{}^t X)\,{}^t g), \tag{84}$$

where $(g, \alpha) \in G_\diamond$ and $(X, Z) \in \mathfrak{g}_\diamond$. And the adjoint representation $\mathrm{ad}_\diamond$ of $\mathfrak{g}_\diamond$ on $\mathrm{End}\,(\mathfrak{g}_\diamond)$ is given by

$$\mathrm{ad}_\diamond((X, Z))((X_1, Z_1)) = [(X, Z), (X_1, Z_1)]_\diamond. \tag{85}$$

We easily see that the Killing form $B_\diamond$ of $\mathfrak{g}_\diamond$ is given by

$$B_\diamond((X_1, Z_1), (X_2, Z_2)) = (m + 4)\,\mathrm{Tr}(X_1X_2). \tag{86}$$

Therefore, the Killing form $B_\diamond$ is highly degenerate.

We let

$$K_\diamond = \{ (k, 0) \in G_\diamond \mid k \in SO(n, \mathbb{R}) \} \cong SO(n, \mathbb{R})$$

be the compact subgroup of $G_\diamond$. Then, the Lie algebra $\mathfrak{k}_\diamond$ of $K_\diamond$ is

$$\mathfrak{k}_\diamond = \{ (X, 0) \in \mathfrak{g}_\diamond \mid X + {}^t X = 0,\ X \in \mathbb{R}^{(n,n)},\ 0 \in \mathbb{R}^{(m,n)} \}.$$

We let $\mathfrak{p}_\diamond$ be the subspace of $\mathfrak{g}_\diamond$ defined by

$$\mathfrak{p}_\diamond = \left\{ (X, Z) \in \mathfrak{g}_\diamond \mid X = {}^t X \in \mathbb{R}^{(n,n)},\ \mathrm{Tr}(X) = 0,\ Z \in \mathbb{R}^{(m,n)} \right\}.$$

Then, we have the following relation:

$$[\mathfrak{k}_\diamond, \mathfrak{k}_\diamond]_\diamond \subset \mathfrak{k}_\diamond \quad \text{and} \quad [\mathfrak{k}_\diamond, \mathfrak{p}_\diamond] \subset \mathfrak{p}_\diamond. \tag{87}$$

In addition, we have

$$\mathfrak{g}_\diamond = \mathfrak{k}_\diamond \oplus \mathfrak{p}_\diamond \quad (\text{direct sum}). \tag{88}$$

We note that the restriction of the Killing form $B_\diamond$ to $\mathfrak{k}_\diamond$ is negative definite and the restriction of $B_\diamond$ to the abelian subalgebra $\mathfrak{r} = \{ (0, Z) \in \mathfrak{g}_\diamond \}$ is identically zero. Since $\mathfrak{r}$ is the radical of $B_\diamond$, $B_\diamond$ is degenerate (see Formula (86)).

An Iwasawa decomposition of the group $SL_{n,m}(\mathbb{R})$ is given by

$$G_\diamond = N_\diamond A_\diamond K_\diamond, \tag{89}$$

where

$$N_\diamond = \left\{ \left(\begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & 1 & \cdots & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, a \right) \in G_\diamond \;\middle|\; a \in \mathbb{R}^{(m,n)} \right\}$$

and

$$A_\diamond = \{ (\mathrm{diag}(a_1, \cdots, a_n), 0) \in G_\diamond \mid a_i \in \mathbb{R}, \prod_{k=1}^{n} a_k = 1, \ 1 \le i \le n \}.$$

An Iwasawa decomposition of the Lie algebra $\mathfrak{g}_\diamond$ of $G_\diamond$ is given by

$$\mathfrak{g}_\diamond = \mathfrak{n}_\diamond + \mathfrak{a}_\diamond + \mathfrak{k}_\diamond, \tag{90}$$

where

$$\mathfrak{n}_\diamond = \left\{ \left(\begin{pmatrix} 0 & * & \cdots & \cdots & * \\ 0 & 0 & \cdots & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, a \right) \in \mathfrak{g}_\diamond \;\middle|\; a \in \mathbb{R}^{(m,n)} \right\}$$

and

$$\mathfrak{a}_\diamond = \{ (\mathrm{diag}(c_1, \cdots, c_n), 0) \in \mathfrak{g}_\diamond \mid c_i \in \mathbb{R}, \sum_{k=1}^{n} c_k = 0, \ \ 1 \le i \le n \}.$$

In fact, $\mathfrak{a}_\diamond$ is the Lie algebra of $A_\diamond$ and $\mathfrak{n}_\diamond$ is the Lie algebra of $N_\diamond$.

Since $\mathrm{Ad}_\diamond(k)\mathfrak{p}_\diamond \subset \mathfrak{p}_\diamond$ for any $k \in K_\diamond$, $K_\diamond$ acts on $\mathfrak{p}_\diamond$ via the adjoint representation of $K_\diamond$ on $\mathfrak{p}_\diamond$ by

$$k_\diamond \cdot (X, Z) = \left(kX\,{}^t k, Z\,{}^t k\right), \tag{91}$$

where $k_\diamond = (k, 0) \in K_\diamond$ with $k \in SO(n, \mathbb{R})$ and $(X, Z) \in \mathfrak{p}_\diamond$.

We put $K_\natural = SO(n, \mathbb{R})$. Action (91) induces the action of $K_\natural$ on the polynomial algebra $\mathrm{Pol}(\mathfrak{p}_\diamond)$ of $\mathfrak{p}_\diamond$ and the symmetric algebra $S(\mathfrak{p}_\diamond)$. We denote by $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$ (resp. $S(\mathfrak{p}_\diamond)^{K_\natural}$) the subalgebra of $\mathrm{Pol}(\mathfrak{p}_\diamond)$ (resp. $S(\mathfrak{p}_\diamond)$) consisting of all $K_\natural$-invariants. The following inner product $(\,,\,)_\diamond$ on $\mathfrak{p}_\diamond$ defined by

$$\left((X_1, Z_1), (X_2, Z_2)\right)_\diamond = \mathrm{Tr}(X_1 X_2) + \mathrm{Tr}(Z_1\,{}^t Z_2), \quad (X_1, Z_1), (X_2, Y_2) \in \mathfrak{p}_\diamond$$

gives an isomorphism as vector spaces

$$\mathfrak{p}_\diamond \cong \mathfrak{p}_\diamond^*, \quad (X, Z) \mapsto f_{X,Z}, \quad (X, Z) \in \mathfrak{p}_\diamond, \tag{92}$$

where $\mathfrak{p}_\diamond^*$ denotes the dual space of $\mathfrak{p}_\diamond$ and $f_{X,Z}$ is the linear functional on $\mathfrak{p}_\diamond$ defined by

$$f_{X,Z}\big((X_1,Z_1)\big) = \big((X,Z),(X_1,Z_1)\big)_\diamond, \quad (X_1,Z_1) \in \mathfrak{p}_\diamond.$$

We let $\mathbb{D}(\mathfrak{P}_{n,m})$ be the algebra of all differential operators on $\mathfrak{P}_{n,m}$ that are invariant under Action (81) of $GL_{n,m}(\mathbb{R})$. It is known that there is a canonical linear bijection of $S(\mathfrak{p}_\diamond)^K$ onto $\mathbb{D}(\mathfrak{P}_{n,m})$. Identifying $\mathfrak{p}_\diamond$ with $\mathfrak{p}_\diamond^*$ by the above Isomorphism (92), we obtain a canonical linear bijection

$$\Psi_{n,m} : \mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural} \longrightarrow \mathbb{D}(\mathfrak{P}_{n,m}) \tag{93}$$

of $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$ onto $\mathbb{D}(\mathfrak{P}_{n,m})$. The map $\Psi_{n,m}$ is described explicitly as follows. We put $N_\diamond = n(n+1)/2 + mn - 1$. We let $\{\varsigma_\alpha \mid 1 \le \alpha \le N_\diamond\}$ be a basis of $\mathfrak{p}_\diamond$. If $P \in \mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$; then,

$$\Big(\Psi_{n,m}(P)f\Big)(gK_\diamond) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \cdot \exp\left(\sum_{\alpha=1}^{N_\diamond} t_\alpha \varsigma_\alpha\right) K_\diamond\right)\right]_{(t_\alpha)=0}, \tag{94}$$

where $f \in C^\infty(\mathfrak{P}_{n,m})$. We refer to [4] (pp. 280–289). In general, it is very hard to express $\Psi_{n,m}(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$.

We take a coordinate (X, Z) in $\mathfrak{p}_\diamond$ such that

$$X = \begin{pmatrix} x_{11} & \frac{1}{2}x_{12} & \dots & \frac{1}{2}x_{1n} \\ \frac{1}{2}x_{12} & x_{22} & \dots & \frac{1}{2}x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}x_{1n} & \frac{1}{2}x_{2n} & \dots & x_{nn} \end{pmatrix} \in \mathfrak{p}_\flat \quad \text{and} \quad Z = (z_{kl}) \in \mathbb{R}^{(m,n)},$$

where

$$\mathfrak{p}_\flat = \{\, X \in \mathbb{R}^{(n,n)} \mid X = {}^tX,\ \mathrm{Tr}(X) = 0 \,\}.$$

We propose the following natural problems.

Problem 10. *Find a complete list of explicit generators of* $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$.

Problem 11. *Find all the relations among a set of generators of* $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$.

Problem 12. *Find an easy or effective way to express the images of the above invariant polynomials under the Helgason map* $\Psi_{n,m}$ *explicitly.*

Problem 13. *Decompose* $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$ *into* $SO(n,\mathbb{R})$*-irreducibles.*

Problem 14. *Find a complete list of explicit generators of the algebra* $\mathbb{D}(\mathfrak{P}_{n,m})$ *or construct explicit* $SL_{n,m}(\mathbb{R})$*-invariant differential operators on* $\mathfrak{P}_{n,m}$.

Problem 15. *Find all the relations among a set of generators of* $\mathbb{D}(\mathfrak{P}_{n,m})$.

Problem 16. *Is* $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$ *finitely generated? Is* $\mathbb{D}(\mathfrak{P}_{n,m})$ *finitely generated?*

Problem 17. *Find the center* $\mathfrak{Z}_{n,m}$ *of* $\mathbb{D}(\mathfrak{P}_{n,m})$.

Problem 18. *Decompose the Hilbert space* $L^2(SL_{n,m}(\mathbb{Z})\backslash SL_{n,m}(\mathbb{R}))$ *into irreducible unitary representations of* $SL_{n,m}(\mathbb{R})$.

8. Invariant Differential Operators on $G^J/(U(n) \times S(m,\mathbb{R}))$

The first part of this section is based on the author's papers [38] and [30] (pp. 285–288).

For two positive integers m and n, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{(\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \ \text{symmetric}\ \}$$

endowed with the following multiplication:

$$(\lambda,\mu;\kappa)\circ(\lambda',\mu';\kappa') = (\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda')$$

with $(\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$. We define the *Jacobi group* G^J of degree n and index m that is the semidirect product of $Sp(2n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J = Sp(2n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law:

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa')\big) = \big(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu';\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu' - \tilde{\mu}\,{}^t\lambda')\big) \tag{95}$$

with $M, M' \in Sp(2n,\mathbb{R}), (\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$, and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)M'$. Then, G^J acts on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\Omega,Z) = \Big(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \tag{96}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$, and $(\Omega,Z) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. We note that the Jacobi group G^J is not a reductive Lie group and the homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is not a symmetric space. From now on, for brevity, we write $\mathbb{H}_{n,m} = \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. The homogeneous space $\mathbb{H}_{n,m}$ is called the Siegel–Jacobi space of degree n and index m.

For a coordinate $(\Omega,Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu})$ and $Z = (z_{kl})$, we write $\Omega = X + i\,Y$ with $X = (x_{ij}),\ Y = (y_{ij})$ real. We put $d\Omega = (d\omega_{ij})$ and $d\overline{\Omega} = (d\overline{\omega}_{ij})$ and set

$$\frac{\partial}{\partial\Omega} = \left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\omega_{ij}}\right) \qquad \text{and} \qquad \frac{\partial}{\partial\overline{\Omega}} = \left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\overline{\omega}_{ij}}\right).$$

We write

$$\begin{aligned} Z &= U + i\,V, \quad U = (u_{kl}), \quad V = (v_{kl}) \ \text{real}, \\ dZ &= (dz_{kl}), \quad d\overline{Z} = (d\overline{z}_{kl}), \end{aligned}$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \ldots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \ldots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \ldots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \ldots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix}.$$

The author proved the following theorems in [39].

Theorem 13. *For any two positive real numbers A and B,*

$$\begin{aligned} ds^2_{n,m;A,B} &= A\cdot\mathrm{Tr}\Big(Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) \\ &\quad + B\,\Big\{\mathrm{Tr}\Big(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) + \mathrm{Tr}\Big(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\Big) \\ &\quad - \mathrm{Tr}\Big(V\,Y^{-1}d\Omega\,Y^{-1}\,{}^t(d\overline{Z})\Big) - \mathrm{Tr}\Big(V\,Y^{-1}d\overline{\Omega}\,Y^{-1}\,{}^t(dZ)\Big)\Big\} \end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under Action (96) of G^J. In fact, $ds^2_{n,m;A,B}$ is a Kähler metric of $\mathbb{H}_{n,m}$.

Proof. See Theorem 1.1 in [39]. □

Theorem 14. *The Laplace operator $\Delta_{m,m;A,B}$ of the G^J-invariant metric $ds^2_{n,m;A,B}$ is given by*

$$\Delta_{n,m;A,B} = \frac{4}{A}\,\mathbb{M}_1 + \frac{4}{B}\mathbb{M}_2, \tag{97}$$

where

$$\begin{aligned}\mathbb{M}_1 \;&=\; \mathrm{Tr}\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \mathrm{Tr}\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) \\ &\quad + \mathrm{Tr}\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \mathrm{Tr}\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right)\end{aligned}$$

and

$$\mathbb{M}_2 = \mathrm{Tr}\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right).$$

Furthermore, $\mathbb{M}_1$ and $\mathbb{M}_2$ are differential operators on $\mathbb{H}_{n,m}$ invariant under Action (96) of G^J.

Proof. See Theorem 1.2 in [39]. □

Remark 15. *Erik Balslev [40] developed the spectral theory of $\Delta_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ for certain arithmetic subgroups of the Jacobi modular group to prove that the set of all eigenvalues of $\Delta_{1,1;1,1}$ satisfies the Weyl law.*

Remark 16. *Yang et al. [41] proved that the scalar curvature of $(\mathbb{H}_{1,1}, ds^2_{1,1;A,B})$ is $-\frac{3}{A}$ and hence is independent of parameter B.*

Remark 17. *The scalar and Ricci curvatures of the Siegel–Jacobi space $(\mathbb{H}_{1,m}, ds^2_{1,m;A,B})$ $(m \geq 1)$ were completely computed by G. Khan and J. Zhang [42] (Proposition 8, pp. 825–826). Furthermore, Khan and Zhang proved that $(\mathbb{H}_{1,m}, ds^2_{1,m;A,B})$ $(m \geq 1)$ has non-negative orthogonal anti-bisectional curvature (cf. [42]) (Proposition 9, p. 826).*

Remark 18. *For an application of the invariant metric $ds^2_{n,m;A,B}$, we refer to [42–44].*

Now, we investigate differential operators on the Siegel–Jacobi space $\mathbb{H}_{n,m}$ invariant under Action (96) of G^J. The stabilizer K^J of G^J at $(iI_n, 0)$ is given by

$$K^J = \left\{ \big(k, (0,0;\kappa)\big) \;\middle|\; k \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \right\},$$

where

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \;\middle|\; A\,{}^tA + B\,{}^tB = I_n,\ A\,{}^tB = B\,{}^tA,\ A, B \in \mathbb{R}^{(n,n)} \right\} \cong U(n).$$

Therefore, $\mathbb{H}_{n,m} \cong G^J/K^J$ is a homogeneous space which is not symmetric. The Lie algebra $\mathfrak{g}^J$ of G^J has a decomposition

$$\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,$$

where

$$\mathfrak{g}^J = \left\{ \big(Z, (P, Q, R)\big) \;\middle|\; Z \in \mathfrak{g},\ P, Q \in \mathbb{R}^{(m,n)},\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\},$$

$$\mathfrak{k}^J = \left\{ \big(X, (0,0,R)\big) \;\middle|\; X \in \mathfrak{k},\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\},$$

$$\mathfrak{p}^J = \left\{ \big(Y, (P,Q,0)\big) \;\middle|\; Y \in \mathfrak{m},\ P,Q \in \mathbb{R}^{(m,n)} \right\}.$$

Here,

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{pmatrix} \;\middle|\; X_1, X_2, X_3 \in \mathbb{R}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\middle|\; {}^tX + X = 0,\ Y = {}^tY \right\}$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \;\middle|\; X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

We note that $\mathfrak{g}$ is the Lie algebra of $Sp(2n,\mathbb{R})$ and $\mathfrak{k}$ is the Lie algebra of $K \cong U(n)$. Thus, the tangent space of the homogeneous space $\mathbb{H}_{n,m}$ at $(iI_n, 0)$ is identified with $\mathfrak{p}^J$.

If $\alpha = \left(\begin{pmatrix} X_1 & Y_1 \\ Z_1 & -{}^tX_1 \end{pmatrix}, (P_1, Q_1, R_1) \right)$ and $\beta = \left(\begin{pmatrix} X_2 & Y_2 \\ Z_2 & -{}^tX_2 \end{pmatrix}, (P_2, Q_2, R_2) \right)$ are elements of $\mathfrak{g}^J$, then the Lie bracket $[\alpha, \beta]$ of α and β is given by

$$[\alpha, \beta] = \left(\begin{pmatrix} X^* & Y^* \\ Z^* & -{}^tX^* \end{pmatrix}, (P^*, Q^*, R^*) \right), \tag{98}$$

where

$$\begin{aligned}
X^* &= X_1X_2 - X_2X_1 + Y_1Z_2 - Y_2Z_1, \\
Y^* &= X_1Y_2 - X_2Y_1 + Y_2\,{}^tX_1 - Y_1\,{}^tX_2, \\
Z^* &= Z_1X_2 - Z_2X_1 + {}^tX_2Z_1 - {}^tX_1Z_2, \\
P^* &= P_1X_2 - P_2X_1 + Q_1Z_2 - Q_2Z_1, \\
Q^* &= P_1Y_2 - P_2Y_1 + Q_2\,{}^tX_1 - Q_1\,{}^tX_2, \\
R^* &= P_1\,{}^tQ_2 - P_2\,{}^tQ_1 + Q_2\,{}^tP_1 - Q_1\,{}^tP_2.
\end{aligned}$$

Lemma 3.

$$[\mathfrak{k}^J, \mathfrak{k}^J] \subset \mathfrak{k}^J, \quad [\mathfrak{k}^J, \mathfrak{p}^J] \subset \mathfrak{p}^J.$$

Proof. The proof follows immediately from Formula (98). □

Lemma 4. *Let*

$$k^J = \left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, (0,0,\kappa) \right) \in K^J$$

with $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)}$ *and*

$$\alpha = \left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P,Q,0) \right) \in \mathfrak{p}^J$$

with $X = {}^tX,\ Y = {}^tY \in \mathbb{R}^{(n,n)},\ P, Q \in \mathbb{R}^{(m,n)}$. *Then, the adjoint action of* K^J *on* $\mathfrak{p}^J$ *is given by*

$$\mathrm{Ad}(k^J)\alpha = \left(\begin{pmatrix} X_* & Y_* \\ Y_* & -X_* \end{pmatrix}, (P_*, Q_*, 0) \right), \tag{99}$$

where

$$\begin{aligned} X_* &= AX\,{}^tA - \big(BX\,{}^tB + BY\,{}^tA + AY\,{}^tB\big), &(100)\\ Y_* &= \big(AX\,{}^tB + AY\,{}^tA + BX\,{}^tA\big) - BY\,{}^tB, &(101)\\ P_* &= P\,{}^tA - Q\,{}^tB, &(102)\\ Q_* &= P\,{}^tB + Q\,{}^tA. &(103)\end{aligned}$$

Proof. We leave the proof to the reader. □

We recall that T_n denotes the vector space of all $n \times n$ symmetric complex matrices. For brevity, we put $T_{n,m} := T_n \times \mathbb{C}^{(m,n)}$. We define the real linear isomorphism $\Phi : \mathfrak{p}^J \longrightarrow T_{n,m}$ by

$$\Phi\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P, Q, 0)\right) = \big(X + i\,Y,\ P + i\,Q\big), \tag{104}$$

where $\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{m}$ and $P, Q \in \mathbb{R}^{(m,n)}$.

We let $S(m,\mathbb{R})$ denote the additive group consisting of all $m \times m$ real symmetric matrices. Now, we define the isomorphism $\theta : K^J \longrightarrow U(n) \times S(m,\mathbb{R})$ by

$$\theta(h, (0,0,\kappa)) = (\delta(h), \kappa), \quad h \in K,\ \kappa \in S(m,\mathbb{R}), \tag{105}$$

where $\delta : K \longrightarrow U(n)$ is the map defined by Formula (49). Identifying $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$, we can identify $\mathfrak{p}^J$ with $T_{n,m} = T_n \times \mathbb{C}^{(m,n)}$.

Theorem 15. *The adjoint representation of K^J on $\mathfrak{p}^J$ is compatible with the natural action of $U(n) \times S(m,\mathbb{R})$ on $T_{n,m}$ defined by*

$$(h,\kappa)\cdot(\omega,z) := (h\,\omega\,{}^th,\ z\,{}^th), \qquad h \in U(n),\ \kappa \in S(m,\mathbb{R}),\ (\omega,z) \in T_{n,m} \tag{106}$$

through maps Φ and θ. Precisely, if $k^J \in K^J$ and $\alpha \in \mathfrak{p}^J$, then we have the following equality:

$$\Phi\big(\mathrm{Ad}(k^J)\alpha\big) = \theta(k^J)\cdot\Phi(\alpha). \tag{107}$$

Here, we regard the complex vector space $T_{n,m}$ as a real vector space.

Proof. The proof can be found in [30] (pp. 286–287). □

We now study algebra $\mathbb{D}(\mathbb{H}_{n,m})$ of all differential operators on $\mathbb{H}_{n,m}$ invariant under the *natural Action* (96) of G^J. Action (106) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}_{n,m} := \mathrm{Pol}\,(T_{n,m})$. We denote by $\mathrm{Pol}_{n,m}^{U(n)}$ the subalgebra of $\mathrm{Pol}_{n,m}$ consisting of all $U(n)$-invariants. Similarly, Action (99) of K induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p}^J)$. We see that through the identification of $\mathfrak{p}^J$ with $T_{n,m}$, the algebra $\mathrm{Pol}(\mathfrak{p}^J)$ is isomorphic to $\mathrm{Pol}_{n,m}$. The following $U(n)$-invariant inner product $(\,,\,)_J$ of the complex vector space $T_{n,m}$ defined by

$$\big((\omega,z),(\omega',z')\big)_J = \mathrm{Tr}\big(\omega\overline{\omega'}\big) + \mathrm{Tr}\big(z\,{}^t\overline{z'}\big), \quad (\omega,z),\ (\omega',z') \in T_{n,m}$$

gives a canonical isomorphism

$$T_{n,m} \cong T_{n,m}^*, \quad (\omega,z) \mapsto f_{\omega,z}, \quad (\omega,z) \in T_{n,m},$$

where $f_{\omega,z}$ is the linear functional on $T_{n,m}$ defined by

$$f_{\omega,z}\big((\omega',z')\big) = \big((\omega',z'),(\omega,z)\big)_{J'}, \quad (\omega',z') \in T_{n,m}.$$

According to Helgason [4] (p. 287), one obtains a canonical linear bijection of $S(T_{n,m})^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$. Here, $S(T_{n,m})$ denotes the symmetric algebra of $T_{n,m}$ and $S(T_{n,m})^{U(n)}$ denotes the subalgebra of all $U(n)$-invariants in $S(T_{n,m})$. Identifying $T_{n,m}$ with $T^*_{n,m}$ by the above isomorphism, one obtains a natural linear bijection

$$\Theta_{n,m} : \mathrm{Pol}_{n,m}^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_{n,m})$$

of $\mathrm{Pol}_{n,m}^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$. The map $\Theta_{n,m}$ is described explicitly as follows. We put $N_* = n(n+1) + 2\,mn$. We let $\{\eta_\alpha \,|\, 1 \le \alpha \le N_*\}$ be a basis of $\mathfrak{p}^J$. If $P \in \mathrm{Pol}\big(\mathfrak{p}^J\big)^K = \mathrm{Pol}_{n,m}^{U(n)}$, then

$$\Big(\Theta_{n,m}(P)f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right)f\left(g\cdot\exp\left(\sum_{\alpha=1}^{N_*} t_\alpha\eta_\alpha\right)K^J\right)\right]_{(t_\alpha)=0}, \tag{108}$$

where $g \in G^J$ and $f \in C^\infty(\mathbb{H}_{n,m})$. In general, it is hard to express $\Theta_{n,m}(P)$ explicitly for a polynomial $P \in \mathrm{Pol}_{n,m}^{U(n)}$.

We propose the following natural problems.

Problem 19. *Find a complete list of explicit generators of* $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 20. *Find all the relations among a set of generators of* $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 21. *Find an easy or effective way to express the images of the above invariant polynomials or generators of* $\mathrm{Pol}_{n,m}^{U(n)}$ *under the Helgason map* $\Theta_{n,m}$ *explicitly.*

Problem 22. *Decompose* $\mathrm{Pol}_{n,m}^{U(n)}$ *into* $U(n)$*-irreducibles.*

Problem 23. *Find a complete list of explicit generators of the algebra* $\mathbb{D}(\mathbb{H}_{n,m})$ *or construct explicit* G^J*-invariant differential operators on* $\mathbb{H}_{n,m}$.

Problem 24. *Find all the relations among a set of generators of* $\mathbb{D}(\mathbb{H}_{n,m})$.

Problem 25. *Is* $\mathrm{Pol}_{n,m}^{U(n)}$ *finitely generated? Is* $\mathbb{D}(\mathbb{H}_{n,m})$ *finitely generated?*

Problem 26. *Are there canonical ways to find generators of* $\mathrm{Pol}_{n,m}^{U(n)}$*?*

Problem 27. *Find the center* $\mathfrak{C}_{n,m}$ *of* $\mathbb{D}(\mathbb{H}_{n,m})$.

We give answers to Problems 19 and 25. We put $\varphi^{(2k)} = \mathrm{Tr}((w\overline{w})^k)$. Moreover, for $1 \le a,b \le m$ and $k \ge 0$, we put

$$\psi_{ba}^{(0,2k,0)} = (\bar{z}(w\overline{w})^k\,{}^t z)_{ba}, \qquad \psi_{ba}^{(1,2k,0)} = (z\overline{w}(w\overline{w})^k\,{}^t z)_{ba},$$
$$\psi_{ba}^{(0,2k,1)} = (\bar{z}(w\overline{w})^k w\,{}^t\bar{z})_{ba}, \qquad \psi_{ba}^{(1,2k,1)} = (z\overline{w}(w\overline{w})^k w\,{}^t\bar{z})_{ba}.$$

Then, we have the following relations:

$$\varphi^{(2k)} = \bar{\varphi}^{(2k)},\ \psi_{ab}^{(1,2k,1)} = \psi_{ba}^{(0,2k+2,0)},\ \psi_{ab}^{(1,2k,0)} = \psi_{ba}^{(1,2k,0)} = \bar{\psi}_{ab}^{(0,2k,1)} = \bar{\psi}_{ba}^{(0,2k,1)}. \tag{109}$$

Minoru Itoh [36] proved the following theorems:

Theorem 16. *The algebra* $\mathrm{Pol}_{n,m}^{U(n)}$ *is generated by the following polynomials*:

$$\varphi^{(2k+2)}, \qquad \mathrm{Re}\,\psi_{ab}^{(0,2k,0)}, \qquad \mathrm{Im}\,\psi_{cd}^{(0,2k,0)}, \qquad \mathrm{Re}\,\psi_{ab}^{(1,2k,0)}, \qquad \mathrm{Im}\,\psi_{ab}^{(1,2k,0)}.$$

Here, the indices run as follows:

$$0 \le k \le n-1, \qquad 1 \le a \le b \le m, \qquad 1 \le c < d \le m.$$

This is seen from the following theorem by using (109):

Theorem 17. *The algebra* $\mathrm{Pol}_{n,m}^{U(n)}$ *is generated by* $\varphi^{(2k+2)}$, $\psi_{ba}^{(0,2k,0)}$, $\psi_{ba}^{(0,2k,1)}$, *and* $\psi_{ba}^{(1,2k,0)}$. *Here, the indices run as follows*:

$$0 \le k \le n-1, \qquad 1 \le a, b \le m.$$

Proof. See Theorem 1.1 in [36]. □

We consider the case $n = m = 1$. For a coordinate (w, ξ) in $T_{1,1} = \mathbb{C} \times \mathbb{C}$, we write $w = r + i\,s,\ \xi = \zeta + i\,\eta \in \mathbb{C},\ r, s, \zeta, \eta$ real. The author of [45] proved that algebra $\mathrm{Pol}_{1,1}^{U(1)}$ is generated by

$$\begin{aligned}
q(w,\xi) &= \frac{1}{4}\,w\,\overline{w} = \frac{1}{4}(r^2+s^2),\\
\alpha(w,\xi) &= \xi\,\overline{\xi} = \zeta^2+\eta^2,\\
\phi(w,\xi) &= \frac{1}{2}\,\mathrm{Re}\,(\xi^2\overline{w}) = \frac{1}{2}\,r(\zeta^2-\eta^2)+s\,\zeta\eta,\\
\psi(w,\xi) &= \frac{1}{2}\,\mathrm{Im}\,(\xi^2\overline{w}) = \frac{1}{2}\,s(\eta^2-\zeta^2)+r\,\zeta\eta.
\end{aligned}$$

In [45], using Formula (108), the author calculated explicitly the images

$$D_1 = \Theta_{1,1}(q), \quad D_2 = \Theta_{1,1}(\alpha), \quad D_3 = \Theta_{1,1}(\phi) \quad \text{and} \quad D_4 = \Theta_{1,1}(\psi)$$

of q, ξ, ϕ, and ψ under the Helgason map $\Theta_{1,1}$. We can show that algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is generated by the following differential operators:

$$\begin{aligned}
D_1 &= y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\\
&\quad +2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right),\\
D_2 &= y\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right),\\
D_3 &= y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right)-2y^2\,\frac{\partial^3}{\partial x\partial u\partial v}\\
&\quad -\left(v\,\frac{\partial}{\partial v}+1\right)D_2
\end{aligned}$$

and

$$\begin{aligned}
D_4 &= y^2\,\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial v^2}-\frac{\partial^2}{\partial u^2}\right)-2\,y^2\,\frac{\partial^3}{\partial y\partial u\partial v}\\
&\quad -v\,\frac{\partial}{\partial u}D_2,
\end{aligned}$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$D_1 D_2 - D_2 D_1 = 2\,y^2 \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) - 4\,y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2\left(v\,\frac{\partial}{\partial v} D_2 + D_2\right).$$

In particular, algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is not commutative. We refer to [45] for more detail.

Hiroyuki Ochiai [46] proved the following results.

Lemma 5. *We have the following relation:*

$$\phi^2 + \psi^2 = q\,\alpha^2.$$

This relation exhausts all the relations among the generators q, α, ϕ, and ψ of $\mathrm{Pol}_{1,1}^{U(1)}$.

Proof. This follows from a direct computation. □

Theorem 18. *We have the following relations:*

(a) $[D_1, D_2] = 2D_3$,
(b) $[D_1, D_3] = 2D_1 D_2 - 2D_3$,
(c) $[D_2, D_3] = -D_2^2$,
(d) $[D_4, D_1] = 0$,
(e) $[D_4, D_2] = 0$,
(f) $[D_4, D_3] = 0$,
(g) $D_3^2 + D_4^2 = D_2 D_1 D_2$.

These seven relations exhaust all the relations among generators D_1, D_2, D_3, and D_4 of $\mathbb{D}(\mathbb{H}_{1,1})$.

Proof. The proof can be found in [46]. □

Finally, we see that for the case when $n = m = 1$, the above eight problems are completely solved.

Remark 19. *According to Theorem 18, we see that D_4 is a generator of the center of $\mathbb{D}(\mathbb{H}_{1,1})$. We observe that the Lapalcian*

$$\Delta_{1,1;A,B} = \frac{4}{A}\,D_1 + \frac{4}{B}\,D_2 \qquad \text{(see Formula (97))}$$

of $(\mathbb{H}_{1,1}, ds^2_{1,1;A,B})$ does not belong to the center of $\mathbb{D}(\mathbb{H}_{1,1})$.

Remark 20. *When $n = 1$ and m is an arbitrary integer, Conley and Raum [47] found the $2m^2 + m + 1$ explicit generators of $\mathbb{D}(\mathbb{H}_{1,m})$ and the explicit one generator of the center of $\mathbb{D}(\mathbb{H}_{1,m})$. They also found the generators of the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}^J)$ of the Jacobi Lie algebra $\mathfrak{g}^J$. The number of generators of the center of $\mathfrak{U}(\mathfrak{g}^J)$ is $1 + \frac{m(m+1)}{2}$.*

We set $\Gamma_n^\flat := Sp(2n, \mathbb{Z})$ and $\Gamma^J := \Gamma_n^\flat \ltimes H_{\mathbb{Z}}^{(n,m)}$ (see notations).

Definition 11. *Function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a* Maass–Jacobi function *for Γ^J if it satisfies the following conditions* (MJ1)–(MJ4)*:*

(MJ1) *f is real analytic;*

(MJ2) $f(\gamma^J \cdot (\Omega, Z)) = f(\Omega, Z)$ for all $\gamma^J \in \Gamma^J$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$;
(MJ3) *f is an eigenfunction of each diffrential operator in the center $\mathfrak{C}_{n,m}$ of $\mathbb{D}(\mathbb{H}_{n,m})$;*
(MJ4) *f has a polynomial growth, i.e., there exist a constant $C > 0$ such that*

$$|f(X + i\,Y, Z)| \leq C\,|p(Y)| \qquad \text{as } \det(Y) \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

We give another notion of Maass–Jacobi functions in the following way.

Definition 12. *We let $\mathbb{D}(\mathbb{H}_{n,m})^*$ be the commutative subalgebra of $\mathbb{D}(\mathbb{H}_{n,m})$ containing the Laplace operator $\Delta_{n,m;A,B}$ of the Siegel–Jacobi space $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$. Function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a* Maass–Jacobi function *for Γ^J with respect to $\mathbb{D}(\mathbb{H}_{n,m})^*$ if it satisfies the following conditions* (MJ1), (MJ2) *and* (MJ4) *together with* $(\mathrm{MJ3})^*$*:*

$(\mathrm{MJ3})^*$ *f is an eigenfunction of each diffrential operator in $\mathbb{D}(\mathbb{H}_{n,m})^*$.*

Function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a weak Maass–Jacobi function *for Γ^J if it satisfies the following conditions,* (MJ1), (MJ2), *and* (MJ4), *together with* $(\mathrm{MJ3})^\diamond$*;*

$(\mathrm{MJ3})^\diamond$ *f is an eigenfunction of the Laplace operator $\Delta_{n,m;A,B}$ of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$.*

We denote by $\mathfrak{W}_{n,m}$ the complex vector space of all Γ^J-invariant real analytic functions on $\mathbb{H}_{n,m}$. We define formally the following inner product:

$$\langle \varphi, \psi \rangle_{n,m} := \int_{\Gamma^J \backslash \mathbb{H}_{n,m}} \varphi(\Omega, Z)\, \overline{\psi(\Omega, Z)}\, dv_{n,m}(\Omega, Z).$$

Here,

$$dv_{n,m}(\Omega, Z) := \det(Y)^{-(n+m+1)} [dX][dY][dU][dV]$$

is a G^J-invariant volume element on $\mathbb{H}_{n,m}$, where for a coordinate $(\Omega, Z) \in \mathbb{H}_{n,m}$ with $\Omega = X + i\,Y$, $X = (x_{ij})$, $Y = (y_{ij})$, $Z = U + i\,V$, $U = (u_{ij})$, $V = (v_{ij})$, X, Y, U, V real,

$$[dX] = \bigwedge_{i \leq j} dx_{ij}, \quad [dY] = \bigwedge_{i \leq j} dy_{ij}, \quad [dU] = \bigwedge_{k,l} du_{kl}, \quad [dV] = \bigwedge_{k,l} dv_{kl}.$$

We let

$$\mathscr{H}_{n,m} := L^2(\Gamma^J \backslash \mathbb{H}_{n,m}) = \{\, \varphi \in \mathfrak{W}_{n,m} \mid \langle \varphi, \varphi \rangle_{n,m} < \infty \,\}$$

be the Hilbert space with the hermitian inner product $\langle\, , \rangle_{n,m}$.

Problem 28. *Develop the theory of harmonic analysis on $\mathscr{H}_{n,m}$ with respect to $\mathbb{D}(\mathbb{H}_{n,m})^*$. In particular, develop the spectral theory of the Laplace operator $\Delta_{n,m;A,B}$ on $\mathscr{H}_{n,m}$.*

Problem 29. *Let $L^2(\Gamma^J \backslash G^J)$ denote the Hilbert space of all real analytic Γ^J-invariant functions on G^J such that*

$$\int_{\Gamma^J \backslash G^J} |f(x)|^2\, d\xi(x) < \infty,$$

where $d\xi(x)$ is a Haar measure on G^J. So we have the hermitian inner product on $L^2(\Gamma^J \backslash G^J)$ defined by

$$\langle f, g \rangle_J := \int_{\Gamma^J \backslash G^J} f(x)\, \overline{g(x)}\, d\xi(x) \qquad (f, g \in L^2(\Gamma^J \backslash G^J)).$$

Decompose the Hilbert space $L^2(\Gamma^J \backslash G^J)$ into irreducible unitary representations of G^J.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Maass, H. Die Bestimmung des Dirichletreihen mit Größencharakteren zu den Modulformen n-ten Grades. *J. Indian Math. Soc.* **1955**, *19*, 1–23.
2. Maass, H. *Siegel Modular Forms and Dirichlet Series*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1971; Volume 216.
3. Selberg, A. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. B* **1956**, *20*, 47–87.
4. Helgason, S. *Groups and Geometric Analysis*; Academic Press: New York, NY, USA, 1984.
5. Brennecken, D.; Ciardo, L.; Hilgert, J. Algebraically independent generators for the algebra of invariant differential operators on $SL_n(\mathbb{R})\backslash SO_n(\mathbb{R})$. *J. Lie Theory* **2021**, *31*, 459–468.
6. Terras, A. *Harmonic Analysis on Symmetric Spaces and Applications II*; Springer: Berlin/Heidelberg, Germany, 1988.
7. Goldfeld, D. *Automorphic Forms and L-Functions for the Group GL(n,R) with an Appendix by Kevin A. Broughan*; Cambridge Studies in Advanced Mathematics; Cambridge University Press: Cambridge, UK, 2006; Volume 99, p. xiv+493, ISBN 978-0-521-83771-2.
8. Shimura, G. Invariant differential operators on hermitian symmetric spaces. *Ann. Math.* **1990**, *132*, 237–272. [CrossRef]
9. Kramer, J.; Mandal, A. Relating Siegel cusp forms to Siegel-Maass forms. *Res. Number Theory* **2022**, *8*, 57. [CrossRef]
10. Harish-Chandra. Representations of a semisimple Lie group on a Banach space. *I. Trans. Am. Math. Soc.* **1953**, *75*, 185–243. [CrossRef]
11. Harish-Chandra. The characters of semisimple Lie groups. *Trans. Am. Math. Soc.* **1956**, *83*, 98–163. [CrossRef]
12. Howe, R. Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. In *The Schur Lectures (1992) (Tel Aviv)*; Israel Mathematics Conference Proceedings; Bar-Ilan University: Ramat Gan, Israel, 1995; Volume 8, pp. 1–182.
13. Weyl, H. *The Classical Groups: Their Invariants and Representations*, 2nd ed.; Princeton University Press: Princeton, NJ, USA, 1946.
14. Minkowski, H. *Gesammelte Abhandlungen*; Chelsea: New York, NY, USA, 1967.
15. Yang, J.-H. Polarized real tori. *J. Korean Math. Soc.* **2015**, *52*, 269–331. [CrossRef]
16. Goresky, M.; Tai, Y.S. The Moduli Space of Real Abelian Varieties with Level Structure. *Compos. Math.* **2003**, *139*, 1–27. [CrossRef]
17. Silhol, R. Compactifications of moduli spaces in real algebraic geometry. *Invent. Math.* **1992**, *107*, 151–202. [CrossRef]
18. Borel, A. Introduction to automorphic forms. *Am. Math. Soc. Provid. RI* **1966**, *9*, 199–210.
19. Borel, A.; Jacquet, H. Automorphic forms and automorphic representations. *Proc. Symposia Pure Math.* **1979**, *33 Pt 1*, 189–202.
20. Grenier, D. On the shape of fundamental domains on $GL(n,\mathbb{R})/O(n)$. *Pac. J. Math.* **1993**, *160*, 53–66. [CrossRef]
21. Grenier, D. Fundamental Domains for the general linear group. *Pac. J. Math.* **1993**, *132*, 293–317. [CrossRef]
22. Grenier, D. An analogue of Siegel's ϕ-operator for automorphic forms for $GL(n,\mathbb{Z})$. *Trans. Am. Math. Soc.* **1992**, *331*, 463–477.
23. Siegel, C.L. The volume of the fundamental domain for some infinite groups. *Trans. Am. Math. Soc.* **1936**, *30*, 209–218. [CrossRef]
24. Garret, P. Volume of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ and $Sp_n(\mathbb{Z})\backslash Sp_n(\mathbb{R})$. Available online: http://www.math.umn.edu/garrett/m/v/volumes.pdf (accessed on 20 April 2014).
25. Borel, A.; Ji, L. Compactifications of symmetric spaces. (English summary). *J. Differ. Geom.* **2007**, *75*, 1–56. [CrossRef]
26. Borel, A.; Ji, L. *Compactifications of Symmetric and Locally Symmetric Spaces*; Mathematics: Theory and Applications; Birkhäuser Boston, Inc.: Boston, MA, USA, 2006; p. xvi+479, ISBN 978-0-8176-3247-2/0-8176-3247-6.
27. Müller, W. Weyl's law for the cuspidal spectrum of SL_n. *Ann. Math.* **2007**, *165*, 275–333. [CrossRef]
28. Lapid, E.; Müller, W. Spectral asymptotics for arithmetic quotients of $SL(n,\mathbb{R})/SO(n,\mathbb{R})$. *Duke Math. J.* **2009**, *149*, 117–155. [CrossRef]
29. Matz, J.; Müller, W. Analytic torsion of arithmetic quotients of the symmetric space $SL(n,\mathbb{R})/SO(n)$. *Geom. Funct. Anal.* **2017**, *27*, 1378–1449. [CrossRef]
30. Yang, J.-H. Geometry and arithmetic on the Siegel-Jacobi space. In *Geometry and Arithmetic on the Siegel-Jacobi Space, Geometry and Analysis on Manifolds, In Memory of Professor Shoshichi Kobayashi (Progress in Mathematics)*; Ochiai, T., Mabuchi, T., Maeda, Y., Noguchi, J., Weinstein, A., Eds.; Birkhäuser: Cham, Switzerland; Springer: Berlin/Heidelberg, Germany; Volume 308, pp. 275–325.
31. Siegel, C.L. Symplectic geometry. *Am. J. Math.* **1943**, *65*, 1–86. [CrossRef]
32. Maass, H. Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen. *Math. Ann.* **1953**, *126*, 44–68. [CrossRef]
33. Siegel, C.L. *Topics in Complex Function Theory: Abelian Functions and Modular Functions of Several Variables*; Wiley-Interscience: New York, NY, USA, 1973; Volume III.

34. Maass, H. *Lectures on Modular Functions of One Complex Variable (Notes by Sunder Lal)*; Tata Institute of Fundamental Research: Bombay, India, 1964; Volume 29, p. iii+262, revised 1983.
35. Yang, J.-H. Invariant differential operators on the Minkowski-Euclid space. *J. Korean Math. Soc.* **2013**, *50*, 275–306. [CrossRef]
36. Itoh, M. *On the Yang Problem*; Max-Planck-Institut für Mathematik: Bonn, Germany, 2012.
37. Yang, J.-H. Singular Jacobi Forms. *Trans. Am. Math. Soc.* **1995**, *347*, 2041–2049. [CrossRef]
38. Yang, J.-H. Problems in the geometry of the Siegel-Jacobi space. *Kyungpook Math. J.* **2024**, *64*, 407–416.
39. Yang, J.-H. Invariant metrics and Laplacians on Siegel-Jacobi space. *J. Number Theory* **2007**, *127*, 83–102. [CrossRef]
40. Balslev, E. *Spectral Theory of the Laplacian on the Modular Jacobi Group Manifold*; Department of Mathematical Sciences, Aarhus University: Aarhus Centrum, Denmark, 2012.
41. Yang, J.-H.; Yong, Y.-H.; Huh, S.-N.; Shin, J.-H.; Min, G.-H. Sectional curvatures of the Siegel-Jacobi space. *Bull. Korean Math. Soc.* **2013**, *50*, 787–799. [CrossRef]
42. Khan, G.; Zhang, J. A hall of statistical mirrors. *Asian J. Math.* **2022**, *26*, 809–846. [CrossRef]
43. Yang, J.; Yin, L. Differential operators for Siegel-Jacobi forms. *Sci. China Math.* **2016**, *59*, 1029–1050. [CrossRef]
44. Ebenfelt, P.; Xiao, M.; Xu, H. Kähler-Einstein metrics and obstruction flatness of circle bundles. *J. Math. Pures Appl*. **2023**, *177*, 368–414. [CrossRef]
45. Yang, J.-H. A note on Maass-Jacobi forms. *Kyungpook Math. J.* **2003**, *43*, 547–566. [CrossRef]
46. Ochiai, H. *A Remark on the Generators of Invariant Differential Operators on Siegel-Jacobi Space of the Smallest Size*; Graduate School of Mathematics, Kyushu University: Fukuoka, Japan, 2022.
47. Conley, C.; Raum, M. Harmonic Maass-Jacobi forms of degree 1 with higher rank indices. *Int. J. Number Theory* **2016**, *12*, 1871–1897. [CrossRef]