**ERRATUM : THE METHOD OF ORBITS FOR REAL LIE GROUPS**

JAE-HYUN YANG

# Erratum

In the article *The Method of Orbits for Real Lie Groups* by Jae-Hyun Yang [Kyungpook Math. J., **42 (2)** (2002), 199–272], **Section 8.2** (pp. 231–237) should be corrected as follows :

### 8.2. The Coadjoint Orbits of Picture

Now we find the coadjoint orbits of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ and describe the connection between the coadjoint orbits and the unitary dual of $H_{\mathbb{R}}^{(g,h)}$ explicitly.

For brevity, we let $G := H_{\mathbb{R}}^{(g,h)}$ as before. Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. We recall that $\mathrm{Sym}(h,\mathbb{R})$ denotes the space of all $h \times h$ real symmetric matrices. We observe that $\mathfrak{g}$ can be regarded as the subalgebra consisting of all $2(g+h) \times 2(g+h)$ real matrices of the form

$$X(\alpha,\beta,\gamma) := \begin{pmatrix} 0 & 0 & 0 & {}^t\beta \\ \alpha & 0 & \beta & \gamma \\ 0 & 0 & 0 & -{}^t\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \alpha,\beta \in \mathbb{R}^{(h,g)}, \ \gamma \in \mathrm{Sym}(h,\mathbb{R})$$

in the Lie algebra $\mathfrak{sp}(2(g+h),\mathbb{R})$ of the symplectic group $Sp(2(g+h),\mathbb{R})$. An easy computation yields

$$[X(\alpha,\beta,\gamma),\ X(\delta,\epsilon,\xi)] = X(0,0,\alpha\,{}^t\epsilon + \epsilon\,{}^t\alpha - \beta\,{}^t\delta - \delta\,{}^t\beta).$$

The dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ can be identified with the vector space consisting of all $2(g+h) \times 2(g+h)$ real matrices of the form

$$F(a,b,c) := \begin{pmatrix} 0 & {}^ta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & {}^tb & 0 & 0 \\ b & c & -a & 0 \end{pmatrix}, \ a,b \in \mathbb{R}^{(h,g)}, \ c \in \mathrm{Sym}(h,\mathbb{R})$$

so that

$$\langle F(a,b,c), X(\alpha,\beta,\gamma)\rangle := \sigma(F(a,b,c)\,X(\alpha,\beta,\gamma)) = 2\,\sigma({}^t\alpha\,a + {}^tb\,\beta) + \sigma(c\,\gamma). \tag{8.24}$$

The adjoint representation $\mathrm{Ad}_G$ of $G$ is given by $\mathrm{Ad}_G(g)X = gXg^{-1}$ for $g \in G$ and $X \in \mathfrak{g}$. For $g \in G$ and $F \in \mathfrak{g}^*$, $gFg^{-1}$ is not of the form $F(a,b,c)$. We denote by $(gFg^{-1})_*$ the

$$\begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} - \text{part}$$

of the matrix $gFg^{-1}$. Then it is easy to see that the coadjoint representation $\mathrm{Ad}_G^*: G \longrightarrow GL(\mathfrak{g}^*)$ is given by $\mathrm{Ad}_G^*(g)F = (gFg^{-1})_*$, where $g \in G$ and $F \in \mathfrak{g}^*$. More precisely,

$$\mathrm{Ad}_G^*(g)F(a,b,c) = F(a + c\mu, b - c\lambda, c), \tag{8.25}$$

where $g = (\lambda, \mu, \kappa) \in G$. So the coadjoint $G$-orbit $\Omega_{a,b}$ at $F(a,b,0) \in \mathfrak{g}^*$ is given by

$$\Omega_{a,b} = \mathrm{Ad}_G^*(G)\, F(a,b,0) = \{F(a,b,0)\}, \text{ a single point.} \tag{8.26}$$

And for any $a, b \in \mathbb{R}^{(h,g)}$ and $c \in \mathrm{Sym}(h,\mathbb{R})$ with $1 \le k = \mathrm{rank}\ c \le h$, the coadjoint $G$-orbit $\Omega_{a,b,c,k}$ at $F(a,b,c) \in \mathfrak{g}^*$ is given by

$$\Omega_{a,b,c,k} = \left\{ F(a + c\mu, b - c\lambda, c) \,\middle|\, \lambda, \mu \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(k,g)} \times \mathbb{R}^{(k,g)}. \tag{8.27}$$

Therefore the coadjoint $G$-orbits in $\mathfrak{g}^*$ fall into two classes:

(**I**) The single points $\left\{ \Omega_{a,b} \,\middle|\, a, b \in \mathbb{R}^{(h,g)} \right\}$ located in the plane $c = 0$.

(**II**) The affine planes $\left\{ \Omega_{a,b,c,k} \,\middle|\, a, b \in \mathbb{R}^{(h,g)},\ c \in \mathrm{Sym}(h,\mathbb{R}),\ 1 \le \mathrm{rank}\ c = k \le h \right\}$.

In other words, the orbit space $\mathcal{O}(G)$ of coadjoint orbits is parametrized by

$$\begin{cases} a, b \in \mathbb{R}^{(h,g)},\ c \in \mathrm{Sym}(h,\mathbb{R}),\ 1 \le k = \mathrm{rank}\, c \le h; \\ (a,b)-\text{plane} \cong \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}. \end{cases}$$

**Definition 8.1.** (a) *The single point coadjoint orbits of the type $\Omega_{a,b}$ are said to be the extremely degenerate $G$-orbits in $\mathfrak{g}^*$.*

(b) *The flat coadjoint orbits of the type $\Omega_{a,b,c,k}$ with $1 \le k = \mathrm{rank}\, c < h$ are said to be the $(h-k)$-degenerate $G$-orbits in $\mathfrak{g}^*$.*

(c) *The flat coadjoint orbits of the type $\Omega_{a,b,c,m}$ with $\mathrm{rank}\, c = m$ are said to be the nondegenerate $G$-orbits in $\mathfrak{g}^*$.*

Since $G$ is a connected and simply connected 2-step nilpotent Lie group, according to A. Kirillov (cf. [47] or [48, Theorem 1, p. 249]), the unitary dual $\widehat{G}$ of $G$ is given by

$$\widehat{G} = \coprod_{a,b,c,k} \Omega_{a,b,c,k} \coprod \left( \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)} \right), \tag{8.28}$$

where $\coprod$ denotes the disjoint union, $a$ and $b$ run over $\mathbb{R}^{(h,g)}$ and $c$ $(1 \le k = \mathrm{rank}\, c \le h)$ runs over $\mathrm{Sym}(h,\mathbb{R})$. We observe that $\Omega_{a,b,c,k} \cong \mathbb{R}^{(k,g)} \times \mathbb{R}^{(k,g)}$. The topology of $\widehat{G}$ may be described as follows. The topology on $\{\Omega_{a,b,c,k} \,|\, a, b \in \mathbb{R}^{(h,g)},\ 0 \neq c \in \mathrm{Sym}(h,\mathbb{R}),\ 1 \le k \le h\}$ is the usual topology of the Euclidean space and the topology on $\{F(a,b,0) \,|\, a, b \in \mathbb{R}^{(h,g)}\}$ is the usual Euclidean topology. But a sequence on $\{\Omega_{a,b,c,k} \,|\, a, b \in \mathbb{R}^{(h,g)},\ 0 \neq c \in \mathrm{Sym}(h,\mathbb{R}),\ 1 \le k \le m\}$ which converges to 0 in the usual topology converges to the whole Euclidean space $\mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)} \times \mathrm{Sym}(h,\mathbb{R})$ in the topology of $\widehat{G}$. This is just the quotient topology on $\mathfrak{g}^*/G$ so that algebraically and topologically $\widehat{G} = \mathfrak{g}^*/G$.

It is well known that each coadjoint orbit is a symplectic manifold. We will state this fact in detail. For the present time being, we fix an element $F$ of $\mathfrak{g}^*$ once and for all. We consider the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$ on $\mathfrak{g}$ defined by

$$\mathbf{B}_F(X,Y) \stackrel{\text{def}}{=} \langle\, F, [X,Y]\,\rangle = \langle\, \mathrm{ad}_{\mathfrak{g}}^*(Y)F, X\,\rangle, \quad X, Y \in \mathfrak{g}, \tag{8.29}$$

where $\mathrm{ad}_{\mathfrak{g}}^* : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}^*)$ denotes the differential of the coadjoint representation $\mathrm{Ad}_G^* : G \longrightarrow GL(\mathfrak{g}^*)$. More precisely, if $F = F(a, b, c),\ X = X(\alpha, \beta, \gamma)$, and $Y = X(\delta, \epsilon, \xi)$, then

$$\mathbf{B}_F(X, Y) = \sigma\{c\,(\alpha\,{}^t\epsilon + \epsilon\,{}^t\alpha - \beta\,{}^t\delta - \delta\,{}^t\beta)\}. \tag{8.30}$$

For $F \in \mathfrak{g}^*$, we let

$$G_F = \{\, g \in G \mid \mathrm{Ad}_G^*(g)F = F \,\}$$

be the stabilizer of the coadjoint action $\mathrm{Ad}^*$ of $G$ on $\mathfrak{g}^*$ at $F$. Since $G_F$ is a closed subgroup of $G$, $G_F$ is a Lie subgroup of $G$. We denote by $\mathfrak{g}_F$ the Lie subalgebra of $\mathfrak{g}$ corresponding to $G_F$. Then it is easy to show that

$$\mathfrak{g}_F = \mathrm{rad}\,\mathbf{B}_F = \left\{\, X \in \mathfrak{g} \mid \mathrm{ad}_{\mathfrak{g}}^*(X)F = 0 \,\right\}. \tag{8.31}$$

Here $\mathrm{rad}\,\mathbf{B}_F$ denotes the radical of $\mathbf{B}_F$ in $\mathfrak{g}$. We let $\dot{\mathbf{B}}_F$ be the non-degenerate alternating $\mathbb{R}$-bilinear form on the quotient vector space $\mathfrak{g}/\mathrm{rad}\ \mathbf{B}_F$ induced from $\mathbf{B}_F$. Since we may identify the tangent space of the coadjoint orbit $\Omega_F \cong G/G_F$ with $\mathfrak{g}/\mathfrak{g}_F = \mathfrak{g}/\mathrm{rad}\,\mathbf{B}_F$, we see that the tangent space of $\Omega_F$ at $F$ is a symplectic vector space with respect to the symplectic form $\dot{\mathbf{B}}_F$.

Now we are ready to prove that the coadjoint orbit $\Omega_F = \mathrm{Ad}_G^*(G)F$ is a symplectic manifold. We denote by $\widetilde{X}$ the smooth vector field on $\mathfrak{g}^*$ associated to $X \in \mathfrak{g}$. That means that for each $\ell \in \mathfrak{g}^*$, we have

$$\widetilde{X}(\ell) = \mathrm{ad}_{\mathfrak{g}}^*(X)\,\ell. \tag{8.32}$$

We define the differential 2-form $B_{\Omega_F}$ on $\Omega_F$ by

$$B_{\Omega_F}(\widetilde{X}, \widetilde{Y}) = B_{\Omega_F}(\mathrm{ad}_{\mathfrak{g}}^*(X)F, \mathrm{ad}_{\mathfrak{g}}^*(Y)F) := \mathbf{B}_F(X, Y), \tag{8.33}$$

where $X, Y \in \mathfrak{g}$.

**Lemma 8.2.** *$B_{\Omega_F}$ is non-degenerate.*

*Proof.* Let $\widetilde{X}$ be the smooth vector field on $\mathfrak{g}^*$ associated to $X \in \mathfrak{g}$ such that $B_{\Omega_F}(\widetilde{X}, \widetilde{Y}) = 0$ for all $\widetilde{Y}$ with $Y \in \mathfrak{g}$. Since $B_{\Omega_F}(\widetilde{X}, \widetilde{Y}) = \mathbf{B}_F(X, Y) = 0$ for all $Y \in \mathfrak{g},\ X \in \mathfrak{g}_F$. Thus $\widetilde{X} = 0$. Hence $B_{\Omega_F}$ is non-degenerate. □

**Lemma 8.2.** *$B_{\Omega_F}$ is closed.*

*Proof.* If $\widetilde{X_1},\ \widetilde{X_2}$, and $\widetilde{X_3}$ are three smooth vector fields on $\mathfrak{g}^*$ associated to $X_1,\ X_2,\ X_3 \in \mathfrak{g}$, then

$$\begin{aligned}
& dB_{\Omega_F}(\widetilde{X_1}, \widetilde{X_2}, \widetilde{X_3}) \\
&= \widetilde{X_1}(B_{\Omega_F}(\widetilde{X_2}, \widetilde{X_3})) - \widetilde{X_2}(B_{\Omega_F}(\widetilde{X_1}, \widetilde{X_3})) + \widetilde{X_3}(B_{\Omega_F}(\widetilde{X_1}, \widetilde{X_2})) \\
&\qquad - B_{\Omega_F}([\widetilde{X_1}, \widetilde{X_2}], \widetilde{X_3}) + B_{\Omega_F}([\widetilde{X_1}, \widetilde{X_3}], \widetilde{X_2}) - B_{\Omega_F}([\widetilde{X_2}, \widetilde{X_3}], \widetilde{X_1}) \\
&= -\langle F, [[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] \rangle \\
&= 0 \qquad \text{(by the Jacobi identity).}
\end{aligned}$$

Therefore $B_{\Omega_F}$ is closed. □

In summary, $(\Omega_F, B_{\Omega_F})$ is a symplectic manifold of dimension $2\,k\,g\ (1 \le k \le h)$ or 0.

In order to describe the irreducible unitary representations of $G$ corresponding to the coadjoint orbits under the Kirillov correspondence, we have to determine the polarizations of $\mathfrak{g}$ for the linear forms $F \in \mathfrak{g}^*$.

*Case* **I**. $F = F(a,b,0),\ a,b \in \mathbb{R}^{(h,g)}$; the extremely degenerate case.

According to (8.26), $\Omega_F = \Omega_{a,b} = \{F(a,b,0)\}$ is a single point. It follows from (8.30) that $\mathbf{B}_F(X,Y) = 0$ for all $X,Y \in \mathfrak{g}$. Thus $\mathfrak{g}$ is the unique polarization of $\mathfrak{g}$ for $F$. The Kirillov correspondence says that the irreducible unitary representation $\pi_{a,b}$ of $G$ corresponding to the coadjoint orbit $\Omega_{a,b}$ is given by

$$\pi_{a,b}(\exp\, X(\alpha,\beta,\gamma)) = e^{2\pi i\, \langle F, X(\alpha,\beta,\gamma)\rangle} = e^{4\pi i\, \sigma(\,{}^t a\alpha + {}^t b\beta)}. \tag{8.34}$$

That is, $\pi_{a,b}$ is a one-dimensional extremely degenerate representation of $G$.

*Case* **II**. $F = F(a,b,c),\ a,b \in \mathbb{R}^{(h,g)},\ c \in \mathrm{Sym}(h,\mathbb{R})$ with $1 \leq \mathrm{rank}\, c \leq h$;

According to (8.27),

$$\Omega_F = \Omega_{a,b,c,k} = \left\{ F(a + c\,\mu, b - c\,\lambda, c) \mid \lambda,\mu \in \mathbb{R}^{(h,g)} \right\}.$$

By (8.30), we see that

$$\mathfrak{k} = \left\{ X(0,\beta,\gamma) \,\middle|\, \beta \in \mathbb{R}^{(m,n)},\ \gamma \in \mathrm{Sym}(h,\mathbb{R}) \right\} \tag{8.35}$$

is a polarization of $\mathfrak{g}$ for $F$, i.e.,$\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ subordinate to $F \in \mathfrak{g}^*$ which is maximal among the totally isotropic vector subspaces of $\mathfrak{g}$ relative to the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$. Let $K$ be the simply connected Lie subgroup of $G$ corresponding to the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. We let

$$\chi_{a,b,c,k;\mathfrak{k}} : K \longrightarrow \mathbb{C}_1^\times$$

be the unitary character of $K$ defined by

$$\chi_{a,b,c,k;\mathfrak{k}}(\exp X(0,\beta,\gamma)) = e^{2\pi i\, \langle F, X(0,\beta,\gamma)\rangle} = e^{2\pi i\, \sigma(c\,\gamma + 2\,\beta\, {}^t b)}. \tag{8.36}$$

The Kirillov correspondence says that the irreducible unitary representation $\pi_{a,b,c,k;\mathfrak{k}}$ of $G$ corresponding to the coadjoint orbit $\Omega_F = \Omega_{a,b,c,k}$ is given by

$$\pi_{a,b,c,k;\mathfrak{k}} = \mathrm{Ind}_K^G\, \chi_{a,b,c,k;\mathfrak{k}}. \tag{8.37}$$

For $a,b,\widetilde{a},\widetilde{b} \in \mathbb{R}^{(h,g)}$, we have

$$\pi_{a,b,c,k;\mathfrak{k}}(\exp X(0,0,\gamma)) = \pi_{\widetilde{a},\widetilde{b},c,k;\mathfrak{k}}(\exp X(0,0,\gamma))$$

for all $\gamma \in \mathrm{Sym}(h,\mathbb{R})$. According to Kirillov's Theorem (cf. [47]), $\pi_{a,b,c,k;\mathfrak{k}}$ is unitarily equivalent to $\pi_{\widetilde{a},\widetilde{b},c,k;\mathfrak{k}}$ for all $a,b,\widetilde{a},\widetilde{b} \in \mathbb{R}^{(h,g)}$. So we denote the equivalence class of $\pi_{a,b,c,k;\mathfrak{k}}$ by $\pi_{c,k;\mathfrak{k}}$. According to Kirillov's Theorem (cf. [47]), we know that the induced representation $\pi_{c,k;\mathfrak{k}}$ is, up to equivalence, independent of the choice of a polarization of $\mathfrak{g}$ for $F$. Thus we denote the equivalence class of $\pi_{c,k;\mathfrak{k}}$ by $\pi_{c,k}$. We see that $\pi_{c,k}$ is realized on the representation space $L^2(\mathbb{R}^{(h,g)}, d\xi)$ as follows:

$$(\pi_{c,k}(g)f)(\xi) = e^{2\pi i\, \sigma\{c(\kappa + \mu\, {}^t\lambda + 2\,\xi\, {}^t\mu)\}} f(\xi + \lambda), \tag{8.38}$$

where $g = (\lambda,\mu,\kappa) \in G$ and $\xi \in \mathbb{R}^{(h,g)}$. Using the fact that

$$\exp X(\alpha,\beta,\gamma) = \left( \alpha, \beta, \gamma + \frac{1}{2}(\alpha\, {}^t\beta - \beta\, {}^t\alpha) \right),$$

we see that $\pi_{c,k}$ is nothing but the Schrödinger representation $U_c = U(\sigma_c)$ of $G$ induced from the one-dimensional unitary representation $\sigma_c$ of $K$ given by $\sigma_c((0,\mu,\kappa)) = e^{2\pi i\,\sigma(c\kappa)}I$. We note that $\pi_{c,k}$ is the representation of $G$ with central character $\chi_{c,k} : \mathfrak{Z} \longrightarrow \mathbb{C}_1^\times$ given by $\chi_{c,k}((0,0,\kappa)) = e^{2\pi i\,\sigma(c\kappa)}$. Here $\mathfrak{Z} = \big\{(0,0,\kappa)\,|\,\kappa \in \mathrm{Sym}(h,\mathbb{R})\,\big\}$ denotes the center of $G$.

It is well known that the monomial representation $\big(\pi_{c,k}, L^2\big(\mathbb{R}^{(h,g)}, d\xi\big)\big)$ of $G$ extends to an operator of trace class

$$\pi^1_{c,k}(\phi) : L^2\big(\mathbb{R}^{(h,g)}, d\xi\big) \longrightarrow L^2\big(\mathbb{R}^{(h,g)}, d\xi\big) \tag{8.39}$$

for all $\phi \in C_c^\infty(G)$. Here $C_c^\infty(G)$ is the vector space of all smooth functions on $G$ with compact support. We let $C_c^\infty(\mathfrak{g})$ and $C(\mathfrak{g}^*)$ the vector space of all smooth functions on $\mathfrak{g}$ with compact support and the vector space of all continuous functions on $\mathfrak{g}^*$ respectively. If $f \in C_c^\infty(\mathfrak{g})$, we define the Fourier cotransform

$$\mathcal{C}F_{\mathfrak{g}} : C_c^\infty(\mathfrak{g}) \longrightarrow C(\mathfrak{g}^*)$$

by

$$\left(\mathcal{C}F_{\mathfrak{g}}(f)\right)(F') := \int_{\mathfrak{g}} f(X)\, e^{2\pi i\,\langle F',X\rangle} dX, \tag{8.40}$$

where $F' \in \mathfrak{g}^*$ and $dX$ denotes the usual Lebesgue measure on $\mathfrak{g}$. According to A. Kirillov (cf. [47]), there exists a measure $\beta$ on the coadjoint orbit

$$\Omega_{a,b,c,k} \approx \mathbb{R}^{(k,g)} \times \mathbb{R}^{(k,g)}$$

$$(a, b \in \mathbb{R}^{(h,g)},\ c \in \mathrm{Sym}(h,\mathbb{R}),\ \mathrm{rank}\, c = k,\ 1 \le k \le h)$$

which is invariant under the coadjoint action of $G$ such that

$$\mathrm{tr}\, \pi^1_{c,k}(\phi) = \int_{\Omega_c} \mathcal{C}F_{\mathfrak{g}}(\phi \circ \exp)(F') d\beta(F') \tag{8.41}$$

holds for all test functions $\phi \in C_c^\infty(G)$, where exp denotes the exponentional mapping of $\mathfrak{g}$ onto $G$. We recall that

$$\pi^1_{c,k}(\phi)(f) := \int_G \phi(x)\,(\pi_{c,k}(x)f)\,dx,$$

where $\phi \in C_c^\infty(G)$ and $f \in L^2(\mathbb{R}^{(m,n)}, d\xi)$. By the Plancherel theorem, the mapping

$$S(G/\mathfrak{Z}) \ni \varphi \longmapsto \pi^1_{c,k}(\varphi) \in \mathrm{TC}(L^2(\mathbb{R}^{(h,g)}, d\xi))$$

extends to a unitary isometry

$$\pi^2_{c,k} : L^2(G/\mathfrak{Z}, \chi_{c,k}) \longrightarrow \mathrm{HS}\big(L^2\big(\mathbb{R}^{(h,g)}, d\xi\big)\big) \tag{8.42}$$

of the representation space $L^2(G/\mathfrak{Z}, \chi_{c,k})$ of $\mathrm{Ind}_{\mathfrak{Z}}^G\, \chi_{c,k}$ onto the complex Hilbert space $\mathrm{HS}\big(L^2\big(\mathbb{R}^{(h,g)}, d\xi\big)\big)$ consisting of all Hilbert-Schmidt operators on $L^2\big(\mathbb{R}^{(h,g)}, d\xi\big)$, where $S(G/\mathfrak{Z})$ is the Schwartz space of all infinitely differentiable complex-valued functions on $G/\mathfrak{Z} \cong \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$ that are rapidly decreasing at infinity and $\mathrm{TC}\big(L^2\big(\mathbb{R}^{(h,g)}, d\xi\big)\big)$ denotes the complex vector space of all continuous $\mathbb{C}$-linear mappings of $L^2\big(\mathbb{R}^{(h,g)}, d\xi\big)$ into itself which are trace class.

In summary, we have the following result.

**Theorem 8.4.** *For $F = F(a,b,0) \in \mathfrak{g}^*$, the irreducible unitary representation $\pi_{a,b}$ of $G$ corresponding to the coadjoint orbit $\Omega_F = \Omega_{a,b}$ under the Kirillov correspondence is an extremely degenerate representation of $G$ given by*

$$\pi_{a,b}\big(\exp X(\alpha,\beta,\gamma)\big) = e^{4\pi i\,\sigma({}^t a\alpha + {}^t b\beta)}.$$

*On the other hand, for $F = F(a,b,c) \in \mathfrak{g}^*$ with $a,b \in \mathbb{R}^{(h,g)}$, $c \in Sym(h,\mathbb{R})$ with $1 \le k = \operatorname{rank} c \le h$, the irreducible unitary representation $\big(\pi_{c,k}, L^2\big(\mathbb{R}^{(h,g)}, d\xi\big)\big)$ of $G$ corresponding to the coadjoint orbit $\Omega_{a,b,c,k}$ under the Kirillov correspondence is unitary equivalent to the Schrödinger representation $\big(U_c, L^2\big(\mathbb{R}^{(h,g)}, d\xi\big)\big)$ and this representation $\pi_{c,k}$ is square integrable modulo its center $\mathfrak{Z}$. For all test functions $\phi \in C_c^\infty(G)$, the character formula*

$$\operatorname{Tr}\big(\pi_{c,k}^2(\phi)\big) = \mathcal{C}(\phi,c)\int_{\operatorname{Sym}(h,\mathbb{R})} \phi(0,0,\kappa)\, e^{2\pi i\,\sigma(c\kappa)} d\kappa$$

*holds for some constant $\mathcal{C}(\phi,c)$ depending on $\phi$ and $c$, where $d\kappa$ is the Lebesgue measure on the Euclidean space* $\operatorname{Sym}(h,\mathbb{R})$.

Now we consider the subgroup $K$ of $G$ given by

$$K := \big\{\, (0,\mu,\kappa) \in G \,\big|\, \mu \in \mathbb{R}^{(h,g)},\ \kappa \in \operatorname{Sym}(h,\mathbb{R}) \,\big\}.$$

The Lie algebra $\mathfrak{k}$ of $K$ is given by (8.35). The dual space $\mathfrak{k}^*$ of $\mathfrak{k}$ may be identified with the space

$$\big\{ F(0,b,c) \,\big|\, b \in \mathbb{R}^{(h,g)},\ c \in \operatorname{Sym}(h,\mathbb{R}) \big\}.$$

We let $\operatorname{Ad}_K^* : K \longrightarrow GL(\mathfrak{k}^*)$ be the coadjoint representation of $K$ on $\mathfrak{k}^*$. The coadjoint $K$-orbit $\omega_{b,c,k}$ at $F(0,b,c) \in \mathfrak{k}^*$ with $k = \operatorname{rank} c$ is given by

$$\omega_{b,c,k} = \operatorname{Ad}_K^*(K)\, F(0,b,c) = \{ F(c\mu, b, c) \mid \mu \in \mathbb{R}^{(h,g)} \}. \tag{8.43}$$

Since $K$ is a commutative group, $[\mathfrak{k},\mathfrak{k}] = 0$ and so the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$ on $\mathfrak{k}$ associated to $F := F(0,b,c)$ identically vanishes on $\mathfrak{k} \times \mathfrak{k}$ (cf. (8.29)). $\mathfrak{k}$ is the unique polarization of $\mathfrak{k}$ for $F = F(0,b,c)$. The Kirillov correspondence says that the irreducible unitary representation $\chi_{b,c}$ of $K$ corresponding to the coadjoint orbit $\omega_{b,c}$ is given by

$$\chi_{b,c}\big(\exp X(0,\beta,\gamma)\big) = e^{2\pi i\,\langle F(0,b,c), X(0,\beta,\gamma)\rangle} = e^{2\pi i\,\sigma(2\, b\, {}^t\beta + c\,\gamma)} \tag{8.44}$$

or

$$\chi_{b,c}((0,\mu,\kappa)) = e^{2\pi i\,\sigma(2\, b\, {}^t\mu + c\,\kappa)}, \quad (0,\mu,\kappa) \in K. \tag{8.45}$$

For $0 \neq c \in \operatorname{Sym}(h,\mathbb{R})$ with $1 \le k = \operatorname{rank} c \le h$, we let $\pi_{c,k}$ be the Schrödinger representation of $G$ given by (8.38). We know that $\pi_{c,k}$ is the irreducible unitary representation of $G$ corresponding to the coadjoint orbit

$$\Omega_{a,b,c,k} = \operatorname{Ad}_G^*(G)\, F(a,b,c) = \Big\{ F(a + c\mu, b - c\lambda, c) \,|\, a,b \in \mathbb{R}^{(h,g)} \Big\}.$$

Let $p : \mathfrak{g}^* \longrightarrow \mathfrak{k}^*$ be the natural projection defined by $p(F(a,b,c)) = F(0,b,c)$. Obviously we have

$$p(\Omega_{a,b,c,k}) = \Big\{ F(0,b,c) \,\Big|\, b \in \mathbb{R}^{(h,g)} \Big\} = \bigcup_{b \in \mathbb{R}^{(h,g)}} \omega_{b,c,k}.$$

According to Kirillov's Theorem (cf. [48, Theorem 1, p. 249], the restriction $\pi_{c,k}|_K$ of $\pi_{c,k}$ to $K$ is the direct integral of all one-dimensional representations $\chi_{b,c}$ of $K$ ($b \in \mathbb{R}^{(h,g)}$). Conversely, we let $\chi_{b,c}$ be the element of $\widehat{K}$ corresponding to the coadjoint orbit $\omega_{b,c,k}$ of

$K$. The induced representation $\mathrm{Ind}_K^G \chi_{b,c}$ is nothing but the Schrödinger representation $\pi_{c,k}$. The coadjoint orbit $\Omega_{a,b,c,k}$ of $G$ is the only coadjoint orbit such that $\Omega_{a,b,c,k} \cap p^{-1}(\omega_{b,c,k})$ is nonempty.

Yang Institute for Advanced Study
Seoul 07989, Korea

and

Department of Mathematics
Inha University
Incheon 22212, Korea
*Email address*: jhyang@inha.ac.kr or yangsiegel@naver.com