ERRATUM: HEISENBERG GROUPS, FUNCTIONS AND THE WEIL REPRESENTATION

JAE-HYUN YANG

Erratum

In the book "*Heisenberg Groups, Functions and the Weil Representation*" by Jae-Hyun Yang [KM Kyung Moon Sa, Seoul, 2012, 155pp. ISBN: 978-89-6105-599-4], **Section 1.7** (pp. 58–64) should be corrected as follows:

1.7 Coadjoint Orbits

In this subsection, we find the coadjoint orbits of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ and describe the connection between the coadjoint orbits and the unitary dual of $H_{\mathbb{R}}^{(n,m)}$ explicitly.

For brevity, we let $G := H_{\mathbb{R}}^{(n,m)}$ as before. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. We recall that $\mathrm{Sym}(m,\mathbb{R})$ denotes the space of all $m \times m$ real symmetric matrices. We observe that $\mathfrak{g}$ can be regarded as the subalgebra consisting of all $2(m+n) \times 2(m+n)$ real matrices of the form

$$X(\alpha,\beta,\gamma) := \begin{pmatrix} 0 & 0 & 0 & {}^t\beta \\ \alpha & 0 & \beta & \gamma \\ 0 & 0 & 0 & -{}^t\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \alpha,\beta \in \mathbb{R}^{(m,n)},\ \gamma \in \mathrm{Sym}(m,\mathbb{R})$$

in the Lie algebra $\mathfrak{sp}(2(m+n),\mathbb{R})$ of the symplectic group $Sp(2(m+n),\mathbb{R})$. An easy computation yields

$$[X(\alpha,\beta,\gamma),\, X(\delta,\epsilon,\xi)] = X(0,0,\alpha\,{}^t\epsilon + \epsilon\,{}^t\alpha - \beta\,{}^t\delta - \delta\,{}^t\beta).$$

The dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ can be identified with the vector space consisting of all $2(m+n) \times 2(m+n)$ real matrices of the form

$$F(a,b,c) := \begin{pmatrix} 0 & {}^ta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & {}^tb & 0 & 0 \\ b & c & -a & 0 \end{pmatrix}, \ a,b \in \mathbb{R}^{(m,n)},\ c \in \mathrm{Sym}(m,\mathbb{R})$$

so that

$$\langle F(a,b,c), X(\alpha,\beta,\gamma)\rangle := \sigma(F(a,b,c)\,X(\alpha,\beta,\gamma)) = 2\,\sigma({}^t\alpha\, a + {}^tb\,\beta) + \sigma(c\,\gamma). \tag{1.7.1}$$

The adjoint representation Ad_G of G is given by $\mathrm{Ad}_G(g)X = gXg^{-1}$ for $g \in G$ and $X \in \mathfrak{g}$. For $g \in G$ and $F \in \mathfrak{g}^*$, gFg^{-1} is not of the form $F(a,b,c)$. We denote by $(gFg^{-1})_*$ the

$$\begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} - \text{part}$$

of the matrix gFg^{-1}. Then it is easy to see that the coadjoint representation $\mathrm{Ad}_G^*: G \longrightarrow GL(\mathfrak{g}^*)$ is given by $\mathrm{Ad}_G^*(g)F = (gFg^{-1})_*$, where $g \in G$ and $F \in \mathfrak{g}^*$. More precisely,

$$\mathrm{Ad}_G^*(g)F(a,b,c) = F(a + c\mu, b - c\lambda, c), \tag{1.7.2}$$

where $g = (\lambda, \mu, \kappa) \in G$. So the coadjoint G-orbit $\Omega_{a,b}$ at $F(a,b,0) \in \mathfrak{g}^*$ is given by

$$\Omega_{a,b} = \mathrm{Ad}_G^*(G)\, F(a,b,0) = \{F(a,b,0)\}, \text{ a single point.} \tag{1.7.3}$$

And for any $a, b \in \mathbb{R}^{(m,n)}$ and $c \in \mathrm{Sym}(m,\mathbb{R})$ with $1 \leq k = \text{rank } c \leq m$, the coadjoint G-orbit $\Omega_{a,b,c,k}$ at $F(a,b,c) \in \mathfrak{g}^*$ is given by

$$\Omega_{a,b,c,k} = \left\{ F(a + c\mu, b - c\lambda, c) \,\middle|\, \lambda, \mu \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(k,n)} \times \mathbb{R}^{(k,n)}. \tag{1.7.4}$$

Therefore the coadjoint G-orbits in $\mathfrak{g}^*$ fall into two classes:

(**I**) The single points $\left\{ \Omega_{a,b} \,\middle|\, a, b \in \mathbb{R}^{(m,n)} \right\}$ located in the plane $c = 0$.

(**II**) The affine planes $\left\{ \Omega_{a,b,c,k} \,\middle|\, a, b \in \mathbb{R}^{(m,n)},\ c \in \mathrm{Sym}(m,\mathbb{R}),\ 1 \leq \text{rank } c = k \leq m \right\}$.

In other words, the orbit space $\mathcal{O}(G)$ of coadjoint orbits is parametrized by

$$\begin{cases} a, b \in \mathbb{R}^{(m,n)},\ c \in \mathrm{Sym}(m,\mathbb{R}),\ 1 \leq k = \mathrm{rank}\, c \leq m; \\ (a,b)-\text{plane} \cong \mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}. \end{cases}$$

Definition. (a) *The single point coadjoint orbits of the type $\Omega_{a,b}$ are said to be the extremely degenerate G-orbits in $\mathfrak{g}^*$.*

(b) *The flat coadjoint orbits of the type $\Omega_{a,b,c,k}$ with $1 \leq k = \mathrm{rank}\, c < m$ are said to be the $(m-k)$-degenerate G-orbits in $\mathfrak{g}^*$.*

(c) *The flat coadjoint orbits of the type $\Omega_{a,b,c,m}$ with $\mathrm{rank}\, c = m$ are said to be the nondegenerate G-orbits in $\mathfrak{g}^*$.*

Since G is a connected and simply connected 2-step nilpotent Lie group, according to A. Kirillov (cf. [16] or [17, Theorem 1, p. 249]), the unitary dual $\widehat{G}$ of G is given by

$$\widehat{G} = \coprod_{a,b,c,k} \Omega_{a,b,c,k} \coprod \left(\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)} \right), \tag{1.7.5}$$

where $\coprod$ denotes the disjoint union, a and b run over $\mathbb{R}^{(m,n)}$ and c $(1 \leq k = \mathrm{rank}\, c \leq m)$ runs over $\mathrm{Sym}(m,\mathbb{R})$. We observe that $\Omega_{a,b,c,k} \cong \mathbb{R}^{(k,n)} \times \mathbb{R}^{(k,n)}$. The topology of $\widehat{G}$ may be described as follows. The topology on $\{\Omega_{a,b,c,k} \mid a, b \in \mathbb{R}^{(m,n)},\ 0 \neq c \in \mathrm{Sym}(m,\mathbb{R}),\ 1 \leq k \leq m\}$ is the usual topology of the Euclidean space and the topology on $\{F(a,b,0) \mid a, b \in \mathbb{R}^{(m,n)}\}$ is the usual Euclidean topology. But a sequence on $\{\Omega_{a,b,c,k} \mid a, b \in \mathbb{R}^{(m,n)},\ 0 \neq c \in \mathrm{Sym}(m,\mathbb{R}),\ 1 \leq k \leq m\}$ which converges to 0 in the usual topology converges to the whole Euclidean space $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)} \times \mathrm{Sym}(m,\mathbb{R})$ in the topology of $\widehat{G}$. This is just the quotient topology on $\mathfrak{g}^*/G$ so that algebraically and topologically $\widehat{G} = \mathfrak{g}^*/G$.

It is well known that each coadjoint orbit is a symplectic manifold. We will state this fact in detail. For the present time being, we fix an element F of $\mathfrak{g}^*$ once and for all. We consider the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$ on $\mathfrak{g}$ defined by

$$\mathbf{B}_F(X,Y) \overset{\text{def}}{=} \langle\, F, [X,Y]\, \rangle = \langle\, \mathrm{ad}_{\mathfrak{g}}^*(Y)F, X\, \rangle, \quad X, Y \in \mathfrak{g}, \tag{1.7.6}$$

where $\mathrm{ad}_{\mathfrak{g}}^* : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}^*)$ denotes the differential of the coadjoint representation $\mathrm{Ad}_G^* : G \longrightarrow GL(\mathfrak{g}^*)$. More precisely, if $F = F(a,b,c),\ X = X(\alpha,\beta,\gamma),$ and $Y = X(\delta,\epsilon,\xi)$, then

$$\mathbf{B}_F(X,Y) = \sigma\{c\,(\alpha\,{}^t\epsilon + \epsilon\,{}^t\alpha - \beta\,{}^t\delta - \delta\,{}^t\beta)\}. \tag{1.7.7}$$

For $F \in \mathfrak{g}^*$, we let

$$G_F = \{\, g \in G \mid \mathrm{Ad}_G^*(g)F = F\,\}$$

be the stabilizer of the coadjoint action Ad^* of G on $\mathfrak{g}^*$ at F. Since G_F is a closed subgroup of G, G_F is a Lie subgroup of G. We denote by $\mathfrak{g}_F$ the Lie subalgebra of $\mathfrak{g}$ corresponding to G_F. Then it is easy to show that

$$\mathfrak{g}_F = \mathrm{rad}\,\mathbf{B}_F = \left\{\, X \in \mathfrak{g} \mid \mathrm{ad}_{\mathfrak{g}}^*(X)F = 0 \,\right\}. \tag{1.7.8}$$

Here $\mathrm{rad}\,\mathbf{B}_F$ denotes the radical of $\mathbf{B}_F$ in $\mathfrak{g}$. We let $\dot{\mathbf{B}}_F$ be the non-degenerate alternating $\mathbb{R}$-bilinear form on the quotient vector space $\mathfrak{g}/\mathrm{rad}\ \mathbf{B}_F$ induced from $\mathbf{B}_F$. Since we may identify the tangent space of the coadjoint orbit $\Omega_F \cong G/G_F$ with $\mathfrak{g}/\mathfrak{g}_F = \mathfrak{g}/\mathrm{rad}\,\mathbf{B}_F$, we see that the tangent space of Ω_F at F is a symplectic vector space with respect to the symplectic form $\dot{\mathbf{B}}_F$.

Now we are ready to prove that the coadjoint orbit $\Omega_F = \mathrm{Ad}_G^*(G)F$ is a symplectic manifold. We denote by $\widetilde{X}$ the smooth vector field on $\mathfrak{g}^*$ associated to $X \in \mathfrak{g}$. That means that for each $\ell \in \mathfrak{g}^*$, we have

$$\widetilde{X}(\ell) = \mathrm{ad}_{\mathfrak{g}}^*(X)\,\ell. \tag{1.7.9}$$

We define the differential 2-form B_{Ω_F} on Ω_F by

$$B_{\Omega_F}(\widetilde{X},\widetilde{Y}) = B_{\Omega_F}(\mathrm{ad}_{\mathfrak{g}}^*(X)F, \mathrm{ad}_{\mathfrak{g}}^*(Y)F) := \mathbf{B}_F(X,Y), \tag{1.7.10}$$

where $X, Y \in \mathfrak{g}$.

Lemma 19. *B_{Ω_F} is non-degenerate.*

Proof. Let $\widetilde{X}$ be the smooth vector field on $\mathfrak{g}^*$ associated to $X \in \mathfrak{g}$ such that $B_{\Omega_F}(\widetilde{X},\widetilde{Y}) = 0$ for all $\widetilde{Y}$ with $Y \in \mathfrak{g}$. Since $B_{\Omega_F}(\widetilde{X},\widetilde{Y}) = \mathbf{B}_F(X,Y) = 0$ for all $Y \in \mathfrak{g},\ X \in \mathfrak{g}_F$. Thus $\widetilde{X} = 0$. Hence B_{Ω_F} is non-degenerate. □

Lemma 20. *B_{Ω_F} is closed.*

Proof. If $\widetilde{X_1},\ \widetilde{X_2},$ and $\widetilde{X_3}$ are three smooth vector fields on $\mathfrak{g}^*$ associated to $X_1,\ X_2,\ X_3 \in \mathfrak{g}$, then

$$\begin{aligned}
&dB_{\Omega_F}(\widetilde{X_1},\widetilde{X_2},\widetilde{X_3})\\
&= \widetilde{X_1}(B_{\Omega_F}(\widetilde{X_2},\widetilde{X_3})) - \widetilde{X_2}(B_{\Omega_F}(\widetilde{X_1},\widetilde{X_3})) + \widetilde{X_3}(B_{\Omega_F}(\widetilde{X_1},\widetilde{X_2}))\\
&\qquad - B_{\Omega_F}([\widetilde{X_1},\widetilde{X_2}],\widetilde{X_3}) + B_{\Omega_F}([\widetilde{X_1},\widetilde{X_3}],\widetilde{X_2}) - B_{\Omega_F}([\widetilde{X_2},\widetilde{X_3}],\widetilde{X_1})\\
&= -\langle F, [[X_1,X_2],X_3] + [[X_2,X_3],X_1] + [[X_3,X_1],X_2]\rangle\\
&= 0 \qquad \text{(by the Jacobi identity).}
\end{aligned}$$

Therefore B_{Ω_F} is closed. □

In summary, (Ω_F, B_{Ω_F}) is a symplectic manifold of dimension $2\,k\,n$ $(1 \le k \le m)$ or 0.

In order to describe the irreducible unitary representations of G corresponding to the coadjoint orbits under the Kirillov correspondence, we have to determine the polarizations of $\mathfrak{g}$ for the linear forms $F \in \mathfrak{g}^*$.

Case **I**. $F = F(a,b,0),\ a,b\in\mathbb{R}^{(m,n)}$; the extremely degenerate case.

According to (1.7.3), $\Omega_F=\Omega_{a,b}=\{F(a,b,0)\}$ is a single point. It follows from (1.7.7) that $\mathbf{B}_F(X,Y)=0$ for all $X,Y\in\mathfrak{g}$. Thus $\mathfrak{g}$ is the unique polarization of $\mathfrak{g}$ for F. The Kirillov correspondence says that the irreducible unitary representation $\pi_{a,b}$ of G corresponding to the coadjoint orbit $\Omega_{a,b}$ is given by

$$\pi_{a,b}(\exp\, X(\alpha,\beta,\gamma)) = e^{2\pi i\,\langle F, X(\alpha,\beta,\gamma)\rangle} = e^{4\pi i\,\sigma(\,{}^t a\alpha+{}^t b\beta)}. \tag{1.7.11}$$

That is, $\pi_{a,b}$ is a one-dimensional extremely degenerate representation of G.

Case **II**. $F=F(a,b,c),\ a,b\in\mathbb{R}^{(m,n)},\ c\in \mathrm{Sym}(m,\mathbb{R})$ with $1\le \mathrm{rank}\, c\le m$;

According to (1.7.4),

$$\Omega_F=\Omega_{a,b,c,k}=\left\{F(a+c\,\mu,\, b-c\,\lambda, c)\,|\,\lambda,\mu\in\mathbb{R}^{(m,n)}\right\}.$$

By (1.7.7), we see that

$$\mathfrak{k}=\left\{\,X(0,\beta,\gamma)\,\middle|\,\beta\in\mathbb{R}^{(m,n)},\ \gamma\in \mathrm{Sym}(m,\mathbb{R})\,\right\} \tag{1.7.12}$$

is a polarization of $\mathfrak{g}$ for F, i.e.,$\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ subordinate to $F\in\mathfrak{g}^*$ which is maximal among the totally isotropic vector subspaces of $\mathfrak{g}$ relative to the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$. Let K be the simply connected Lie subgroup of G corresponding to the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. We let

$$\chi_{a,b,c,k;\mathfrak{k}}: K\longrightarrow \mathbb{C}_1^\times$$

be the unitary character of K defined by

$$\chi_{a,b,c,k;\mathfrak{k}}(\exp X(0,\beta,\gamma)) = e^{2\pi i\,\langle F,X(0,\beta,\gamma)\rangle} = e^{2\pi i\,\sigma(c\,\gamma+2\,\beta\,{}^t b)}. \tag{1.7.13}$$

The Kirillov correspondence says that the irreducible unitary representation $\pi_{a,b,c,k;\mathfrak{k}}$ of G corresponding to the coadjoint orbit $\Omega_F=\Omega_{a,b,c,k}$ is given by

$$\pi_{a,b,c,k;\mathfrak{k}} = \mathrm{Ind}_K^G\,\chi_{a,b,c,k;\mathfrak{k}}. \tag{1.7.14}$$

For $a,b,\widetilde{a},\widetilde{b}\in\mathbb{R}^{(m,n)}$, we have

$$\pi_{a,b,c,k;\mathfrak{k}}(\exp X(0,0,\gamma)) = \pi_{\widetilde{a},\widetilde{b},c,k;\mathfrak{k}}(\exp X(0,0,\gamma))$$

for all $\gamma\in\mathrm{Sym}(m,\mathbb{R})$. According to Kirillov's Theorem (cf. [16]), $\pi_{a,b,c,k;\mathfrak{k}}$ is unitarily equivalent to $\pi_{\widetilde{a},\widetilde{b},c,k;\mathfrak{k}}$ for all $a,b,\widetilde{a},\widetilde{b}\in\mathbb{R}^{(m,n)}$. So we denote the equivalence class of $\pi_{a,b,c,k;\mathfrak{k}}$ by $\pi_{c,k;\mathfrak{k}}$. According to Kirillov's Theorem (cf. [16]), we know that the induced representation $\pi_{c,k;\mathfrak{k}}$ is, up to equivalence, independent of the choice of a polarization of $\mathfrak{g}$ for F. Thus we denote the equivalence class of $\pi_{c,k;\mathfrak{k}}$ by $\pi_{c,k}$. We see that $\pi_{c,k}$ is realized on the representation space $L^2(\mathbb{R}^{(m,n)},d\xi)$ as follows:

$$(\pi_{c,k}(g)f)(\xi) = e^{2\pi i\,\sigma\{c(\kappa+\mu\,{}^t\lambda+2\,\xi\,{}^t\mu)\}}f(\xi+\lambda), \tag{1.7.15}$$

where $g=(\lambda,\mu,\kappa)\in G$ and $\xi\in\mathbb{R}^{(m,n)}$. Using the fact that

$$\exp X(\alpha,\beta,\gamma) = \left(\alpha,\beta,\gamma+\frac{1}{2}\big(\alpha\,{}^t\beta-\beta\,{}^t\alpha\big)\right),$$

we see that $\pi_{c,k}$ is nothing but the Schrödinger representation $U_c=U(\sigma_c)$ of G induced from the one-dimensional unitary representation σ_c of K given by $\sigma_c((0,\mu,\kappa)) = e^{2\pi i\,\sigma(c\,\kappa)}I$ (cf. (1.4.6) and (1.4.8)). We note that $\pi_{c,k}$ is the representation of G with central character

$\chi_{c,k} : \mathfrak{Z} \longrightarrow \mathbb{C}_1^\times$ given by $\chi_{c,k}((0,0,\kappa)) = e^{2\pi i\, \sigma(c\kappa)}$. Here $\mathfrak{Z} = \big\{(0,0,\kappa) \mid \kappa \in \mathrm{Sym}(m,\mathbb{R})\big\}$ denotes the center of G.

It is well known that the monomial representation $\big(\pi_{c,k}, L^2\big(\mathbb{R}^{(m,n)}, d\xi\big)\big)$ of G extends to an operator of trace class

$$\pi^1_{c,k}(\phi) : L^2\big(\mathbb{R}^{(m,n)}, d\xi\big) \longrightarrow L^2\big(\mathbb{R}^{(m,n)}, d\xi\big) \tag{1.7.16}$$

for all $\phi \in C_c^\infty(G)$. Here $C_c^\infty(G)$ is the vector space of all smooth functions on G with compact support. We let $C_c^\infty(\mathfrak{g})$ and $C(\mathfrak{g}^*)$ the vector space of all smooth functions on $\mathfrak{g}$ with compact support and the vector space of all continuous functions on $\mathfrak{g}^*$ respectively. If $f \in C_c^\infty(\mathfrak{g})$, we define the Fourier cotransform

$$\mathcal{C}F_{\mathfrak{g}} : C_c^\infty(\mathfrak{g}) \longrightarrow C(\mathfrak{g}^*)$$

by

$$\left(\mathcal{C}F_{\mathfrak{g}}(f)\right)(F') := \int_{\mathfrak{g}} f(X)\, e^{2\pi i\,\langle F', X\rangle} dX, \tag{1.7.17}$$

where $F' \in \mathfrak{g}^*$ and dX denotes the usual Lebesgue measure on $\mathfrak{g}$. According to A. Kirillov (cf. [47]), there exists a measure β on the coadjoint orbit

$$\Omega_{a,b,c,k} \approx \mathbb{R}^{(k,n)} \times \mathbb{R}^{(k,n)}$$

$$(a, b \in \mathbb{R}^{(m,n)},\ c \in \mathrm{Sym}(m,\mathbb{R}),\ \mathrm{rank}\, c = k,\ 1 \le k \le m)$$

which is invariant under the coadjoint action of G such that

$$\mathrm{tr}\, \pi^1_{c,k}(\phi) = \int_{\Omega_c} \mathcal{C}F_{\mathfrak{g}}(\phi \circ \exp)(F') d\beta(F') \tag{1.7.18}$$

holds for all test functions $\phi \in C_c^\infty(G)$, where exp denotes the exponentional mapping of $\mathfrak{g}$ onto G. We recall that

$$\pi^1_{c,k}(\phi)(f) := \int_G \phi(x)\, (\pi_{c,k}(x) f)\, dx,$$

where $\phi \in C_c^\infty(G)$ and $f \in L^2(\mathbb{R}^{(m,n)}, d\xi)$. By the Plancherel theorem, the mapping

$$S(G/\mathfrak{Z}) \ni \varphi \longmapsto \pi^1_{c,k}(\varphi) \in \mathrm{TC}(L^2(\mathbb{R}^{(m,n)}, d\xi))$$

extends to a unitary isometry

$$\pi^2_{c,k} : L^2(G/\mathfrak{Z}, \chi_{c,k}) \longrightarrow \mathrm{HS}\big(L^2\big(\mathbb{R}^{(m,n)}, d\xi\big)\big) \tag{1.7.19}$$

of the representation space $L^2(G/\mathfrak{Z}, \chi_{c,k})$ of $\mathrm{Ind}_{\mathfrak{Z}}^G\, \chi_{c,k}$ onto the complex Hilbert space $\mathrm{HS}\big(L^2\big(\mathbb{R}^{(m,n)}, d\xi\big)\big)$ consisting of all Hilbert-Schmidt operators on $L^2\big(\mathbb{R}^{(m,n)}, d\xi\big)$, where $S(G/\mathfrak{Z})$ is the Schwartz space of all infinitely differentiable complex-valued functions on $G/\mathfrak{Z} \cong \mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ that are rapidly decreasing at infinity and $\mathrm{TC}\big(L^2\big(\mathbb{R}^{(m,n)}, d\xi\big)\big)$ denotes the complex vector space of all continuous $\mathbb{C}$-linear mappings of $L^2\big(\mathbb{R}^{(m,n)}, d\xi\big)$ into itself which are trace class.

In summary, we have the following result.

Theorem 9. *For $F = F(a,b,0) \in \mathfrak{g}^*$, the irreducible unitary representation $\pi_{a,b}$ of G corresponding to the coadjoint orbit $\Omega_F = \Omega_{a,b}$ under the Kirillov correspondence is an extremely degenerate representation of G given by*

$$\pi_{a,b}\big(\exp X(\alpha,\beta,\gamma)\big) = e^{4\pi i\, \sigma({}^t a\alpha + {}^t b\beta)}.$$

On the other hand, for $F = F(a,b,c) \in \mathfrak{g}^*$ *with* $a, b \in \mathbb{R}^{(m,n)}$, $c \in Sym(m,\mathbb{R})$ *with* $1 \leq k = \operatorname{rank} c \leq m$, *the irreducible unitary representation* $\big(\pi_{c,k}, L^2\big(\mathbb{R}^{(m,n)}, d\xi\big)\big)$ *of* G *corresponding to the coadjoint orbit* $\Omega_{a,b,c,k}$ *under the Kirillov correspondence is unitary equivalent to the Schrödinger representation* $\big(U_c, L^2\big(\mathbb{R}^{(m,n)}, d\xi\big)\big)$ *and this representation* $\pi_{c,k}$ *is square integrable modulo its center* $\mathfrak{Z}$. *For all test functions* $\phi \in C_c^\infty(G)$, *the character formula*

$$\operatorname{Tr}\big(\pi_{c,k}^2(\phi)\big) = \mathcal{C}(\phi, c) \int_{\operatorname{Sym}(m,\mathbb{R})} \phi(0,0,\kappa)\, e^{2\pi i\, \sigma(c\kappa)} d\kappa$$

holds for some constant $\mathcal{C}(\phi, c)$ *depending on* ϕ *and* c, *where* $d\kappa$ *is the Lebesgue measure on the Euclidean space* $\operatorname{Sym}(m,\mathbb{R})$.

Now we consider the subgroup K of G (cf. (1.4.1)) given by

$$K := \big\{\, (0,\mu,\kappa) \in G \,\big|\, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \operatorname{Sym}(m,\mathbb{R}) \,\big\}.$$

The Lie algebra $\mathfrak{k}$ of K is given by (1.7.12). The dual space $\mathfrak{k}^*$ of $\mathfrak{k}$ may be identified with the space

$$\big\{ F(0,b,c) \,\big|\, b \in \mathbb{R}^{(m,n)},\, c \in \operatorname{Sym}(m,\mathbb{R}) \big\}.$$

We let $\operatorname{Ad}_K^* : K \longrightarrow GL(\mathfrak{k}^*)$ be the coadjoint representation of K on $\mathfrak{k}^*$. The coadjoint K-orbit $\omega_{b,c,k}$ at $F(0,b,c) \in \mathfrak{k}^*$ with $k = \operatorname{rank} c$ is given by

$$\omega_{b,c,k} = \operatorname{Ad}_K^*(K)\, F(0,b,c) = \{ F(c\mu, b, c) \mid \mu \in \mathbb{R}^{(m,n)} \}. \tag{1.7.20}$$

Since K is a commutative group, $[\mathfrak{k},\mathfrak{k}] = 0$ and so the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$ on $\mathfrak{k}$ associated to $F := F(0,b,c)$ identically vanishes on $\mathfrak{k} \times \mathfrak{k}$ (cf. (1.7.6)). $\mathfrak{k}$ is the unique polarization of $\mathfrak{k}$ for $F = F(0,b,c)$. The Kirillov correspondence says that the irreducible unitary representation $\chi_{b,c}$ of K corresponding to the coadjoint orbit $\omega_{b,c}$ is given by

$$\chi_{b,c}\big(\exp X(0,\beta,\gamma)\big) = e^{2\pi i\, \langle F(0,b,c), X(0,\beta,\gamma)\rangle} = e^{2\pi i\, \sigma(2\, b\, {}^t\beta + c\,\gamma)} \tag{1.7.21}$$

or

$$\chi_{b,c}((0,\mu,\kappa)) = e^{2\pi i\, \sigma(2\, b\, {}^t\mu + c\,\kappa)}, \quad (0,\mu,\kappa) \in K. \tag{1.7.22}$$

For $0 \neq c \in \operatorname{Sym}(m,\mathbb{R})$ with $1 \leq k = \operatorname{rank} c \leq m$, we let $\pi_{c,k}$ be the Schrödinger representation of G given by (1.7.15). We know that $\pi_{c,k}$ is the irreducible unitary representation of G corresponding to the coadjoint orbit

$$\Omega_{a,b,c,k} = \operatorname{Ad}_G^*(G)\, F(a,b,c) = \Big\{ F(a + c\mu, b - c\lambda, c) \,|\, a, b \in \mathbb{R}^{(m,n)} \Big\}.$$

Let $p : \mathfrak{g}^* \longrightarrow \mathfrak{k}^*$ be the natural projection defined by $p(F(a,b,c)) = F(0,b,c)$. Obviously we have

$$p(\Omega_{a,b,c,k}) = \Big\{ F(0,b,c) \,\Big|\, b \in \mathbb{R}^{(m,n)} \Big\} = \bigcup_{b \in \mathbb{R}^{(m,n)}} \omega_{b,c,k}.$$

According to Kirillov's Theorem (cf. [17, Theorem 1, p. 249], the restriction $\pi_{c,k}|_K$ of $\pi_{c,k}$ to K is the direct integral of all one-dimensional representations $\chi_{b,c}$ of K ($b \in \mathbb{R}^{(m,n)}$). Conversely, we let $\chi_{b,c}$ be the element of $\widehat{K}$ corresponding to the coadjoint orbit $\omega_{b,c,k}$ of K. The induced representation $\operatorname{Ind}_K^G \chi_{b,c}$ is nothing but the Schrödinger representation $\pi_{c,k}$. The coadjoint orbit $\Omega_{a,b,c,k}$ of G is the only coadjoint orbit such that $\Omega_{a,b,c,k} \cap p^{-1}(\omega_{b,c,k})$ is nonempty.

Yang Institute for Advanced Study
Seoul 07989, Korea

AND

DEPARTMENT OF MATHEMATICS
INHA UNIVERSITY
INCHEON 22212, KOREA
Email address: jhyang@inha.ac.kr or yangsiegel@naver.com